BITS AND PIECES ON CLIFFORD ALGEBRAS

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ABSTRACT

This is supposed to give an introduction into the theory of Clifford algebras, their basic properties and the relations between Clifford algebras in different dimensions and with respect to different quadratic forms. It also features many examples and hopefully is a good foundation for proceeding to the definition and theory of Spin groups and spinor bundles.

1 PRELIMINARIES AND NOTATION

To save time later, I will fix notation common to all sections now and use it without further notice. K denotes a field. Vector spaces V and associative algebras A with 1 are over K. As we are dealing with bilinear forms, special care has to be taken if K is of prime characteristic (particularly 2) but I shall not mention these cases as they don't cause major problems and we are mainly concerned with $\mathbf{R}$ for geometric purposes anyway. Once we come to examples, K will be $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, $K(n)$ will denote $n \times n$ matrices with entries in K and $\otimes$ will be short for $\otimes_{\mathbf{R}}$.

$q : V \to K$ will be a quadratic form and $b : V \times V \to K$ the symmetric bilinear form associated to q by polarisation. $\{e_1, \ldots, e_n\}$ will denote an ortho'normal' basis of V such that the matrix of b with respect to this basis is the one we get in Sylvester's theorem, i.e. we have $r, s \in \mathbf{N}$ with $r + s \leq n$ such that $q(e_i) = 1$ for $i \leq r$, $q(e_i) = -1$ for $r < i \leq r + s$ and $q(e_i) = 0$ for $r + s < i$. If q is non-degenerate with r and s as above, we shall write $q_{r,s}$ for the quadratic form.

We will also use multi-indices I where $I = (i_1, \ldots, i_k)$ is an ordered subset of $\{1, \ldots, n\}$. e_I then denotes the product $e_{i_1} \cdot \ldots \cdot e_{i_k}$. We define $e_\emptyset$ to be 1.

Occasionally sums $v = \sum_{i=1}^n v_i e_i$ will be denoted by $v = v_i e_i$ if the range of the summation is the obvious one.

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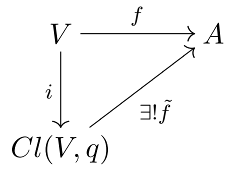
2 UNIVERSAL PROPERTY

A Clifford algebra can be defined by its universal property: Given (V, q) a vector space with a quadratic form and A an associative algebra with 1 then a *Clifford algebra of* (V, q) is an associative algebra with 1, $Cl(V, q)$, such that we have an inclusion $i : V \hookrightarrow Cl(V, q)$ and any linear map $f : V \to A$ with



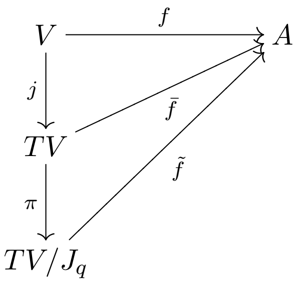
$$(*) \qquad f(v)^2 = -q(v)1$$

extends uniquely to an algebra-homomorphism $\tilde{f} : Cl(V, q) \to A$. If it is clear which (V, q) is used, we only write $Cl(V)$ instead of $Cl(V, q)$ and for the quadratic forms $q_{r,s}$, we define $Cl_{r,s}$ to be $Cl(V, q_{r,s})$. Also, with $v \in V$ and $\lambda \in K$, $i(v)$ and $\lambda 1$ will frequently be written as v and λ only.

3 EXISTENCE AND UNIQUENESS

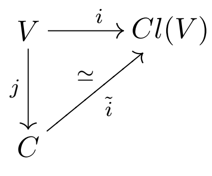
EXISTENCE To ensure existence of a Clifford algebra for each (V, q) we explicitly construct an algebra satisfying the universal property. Keeping in mind this construction will be useful for establishing properties of Clifford algebras and computing examples in sections 4 and 5.

Consider TV, the tensor algebra of V, and the ideal J_q generated by elemens of the form $v \otimes v + q(v)1$ with $v \in V$. Let $f : V \to A$ be a linear map from V into an associative algebra with 1, A, satisfying $(*)$. f can be extended to an algebra-homomorphism $\bar{f} : TV \to A$ using the usual inclusion $j : V \hookrightarrow TV$. Since $J_q \subset \ker \bar{f}$, $\bar{f}$ descends to an algebra-homomorphism $\tilde{f} : TV/J_q \to A$. Also, $J_q \cap j(V) = 0$ and hence $i = \pi j : V \to TV/J_q$ is still an inclusion. Furthermore $i(V)$ and 1 generate all of TV/J_q.



Hence commutativity of the diagram and the axioms for algebra homomorphisms uniquely determine $\tilde{f}$ and TV/J_q satisfies the universal property.

UNIQUENESS Noting that elements of the form $i(v)^2 + q(v)1$ are equal to 0 in $Cl(V)$, we see that i satisfies $(*)$. Now let C be another associative algebra with 1 satisfying the universal property. Then $i : V \hookrightarrow Cl(V)$ descends to a unique algebra-homomorphism $\tilde{i} : C \to Cl(V)$ that is a vector-space-isomorphism $j(V) \simeq i(V)$. So in particular, j also satisfies $(*)$ and thus exchanging C and $Cl(V)$ in the diagram and repeating the same argument for j proves that $\tilde{i}$ is an isomorphism of algebras.



Thus $Cl(V)$ is unique up to isomorphism and it is justified to speak of *the* Clifford algebra of (V, q). In particular, $Cl(V) \simeq TV/J_q$.

4 PROPERTIES OF CLIFFORD ALGEBRAS

FUNCTORIALITY Cl is a functor from the category of vector spaces with quadratic forms and linear maps $\varphi : (V, q_V) \to (W, q_W)$ compatible with the quadratic form, i.e. $\varphi^* q_W = q_V$, to the category of associative algebras with 1: With φ as above, by the universal property, $j\varphi$ extends uniquely to a homomorphism $\widetilde{j\varphi}$ of Clifford algebras which will be $Cl(\varphi)$. The uniqueness of this extension ensures that Cl respects composition and preserves identities.

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ Cl(V) & \xrightarrow{\widetilde{j\varphi}} & Cl(W) \end{array}$$

IDENTITIES Making use of $(*)$, we can see that for $v, w \in i(V)$, $(v+w)^2 = -q(v+w) = -q(v) - 2b(v,w) - q(w) = v^2 - 2b(v,w) + w^2$. On the other hand we have $(v+w)^2 = v^2 + vw + wv + w^2$ and thus we get

$$vw + wv = -2b(v, w).$$

Applying this to a basis as described in section 1 gives

$$e_i e_j = -e_j e_i \quad \text{for } i \neq j \qquad \text{and} \qquad e_i^2 = -1, 1, 0.$$

BASIS Now we can find a basis for the Clifford algebra. Choosing $\{e_1, \ldots, e_n\}$ as above as a basis for V, we recall the construction of the Clifford algebra as a quotient of the tensor algebra. As elements of the form $e_{j_1} \otimes \ldots \otimes e_{j_k}$ (with $j_1, \ldots, j_k$ not necessarily ordered) together with 1 form a basis of the tensor algebra, elements of the form $e_{j_1} \ldots e_{j_k}$ together with 1 will span the Clifford algebra.

Taking into account the relations given by the two equations above, we see that sets of the forms $\{e_i e_j, e_j e_i\}$ and $\{1, e_i^2\}$ are linearly dependent and thus the Clifford algebra is spanned by elements of the form e_I where I is an increasing subset of $\{1, \ldots, n\}$. Due to the restrictions in the choice of I, there are no further relations between the e_I and $\{e_I\}$ is a basis of the Clifford algebra.

Given this basis of the Clifford algebra, it is immediate that there is an isomorphism of vector spaces $Cl(V) \simeq \Lambda^* V$ mapping e_I with respect to Clifford-multiplication to e_I with respect to the wedge product. This is not in general an isomorphism of algebras, as can be seen by looking at the products $e_i e_i$ in $Cl(V)$ and $e_i \wedge e_i$ in $\Lambda^* V$. In fact it is only an isomorphism of algebras if $q = 0$. Thus, for nontrivial q, the Clifford algebra gives a genuinely new algebra structure on $\mathbf{R}^{2^n}$.

GRADINGS We say an element v of $Cl(V)$ is of *degree* k if it is of the form $v = \sum_{|I|=k} a_I e_I$. This is well-defined: Consider orthonormal bases $\{e_1, \ldots, e_n\}$ and $\{e'_1, \ldots, e'_n\}$ with $e'_i = o(e_i) = o_{ij} e_j$ where $o \in \mathrm{O}(V, q)$.

By functoriality, this gives an algebra-isomorphism $Cl(o) : Cl(V) \to Cl(V)$. $Cl(o)$ takes generators e_i to generators e'_i and thus we have $o(\sum_{|I|=k} a_I e_I) = \sum_{|I|=k} a_I e'_I$, proving that the degree is preserved.

We will write $Cl^k(V)$ for the subspace of elements of degree k in $Cl(V)$. Having in mind the known grading of $\Lambda^* V$ and the isomorphism mentioned in the previous paragraph, it follows that this isomorphism is degree-preserving and thus we have a corresponding grading on $Cl(V)$:

$$Cl(V) = \bigoplus_{k=0}^{n} Cl^k(V)$$

Note that this is just a direct sum of vector spaces and whilst the algebra structure on $\Lambda^* V$ is also a graded ring, the algebra structure on $Cl(V)$ isn't in general. This grading also gives rise to coarser grading of $Cl(V)$ into its *even part* $Cl^+(V) = \bigoplus_{k \text{ even}} Cl^k(V)$ and its *odd part* $Cl^-(V) = \bigoplus_{k \text{ odd}} Cl^k(V)$. $Cl^+(V)$ is a subalgebra of $Cl(V)$ and this $\mathbf{Z}_2$ grading has the structure of a graded ring.

CENTRE It is not difficult to compute the centre of $Cl(V)$. By definition, an element v is in $Z(Cl(V))$ if it commutes with all other elements. It is enough to check this for $v = e_I$ and since $Cl(V)$ is generated by the e_i,we have that $e_I \in Z(Cl(V))$ if $e_I e_j = e_j e_I$ for all $1 \leq j \leq n$. However, we also know that

$$e_I e_j = (-1)^{|I|-1} e_j e_I \quad \text{for} \quad j \in I \quad \text{and} \quad e_I e_j = (-1)^{|I|} e_j e_I \quad \text{for} \quad j \notin I.$$

As e_I has to commute with all of the e_j, both conditions can only be satisfied if $|I| \equiv |I| - 1 \pmod 2$, which is impossible, or if either of the two conditions is vacuous. The latter is the case for $|I| = 0$ and $|I| = n$. For $|I| = 0$ we have $e_I = 1$ which is always in the centre. For $|I| = n$ we are left with the equation $e_I e_j = (-1)^{n-1} e_j e_I$ with $j \in I$, i.e e_I is in the centre only if n is odd. Thus, as a result we get:

$$Z(Cl(V)) = \begin{cases} Cl^0 & \text{if } n \text{ is even} \\ Cl^0 \oplus Cl^n & \text{if } n \text{ is odd} \end{cases}.$$

INVOLUTIONS Finally, we define two involutions on $Cl(V)$: Firstly let α be the algebra-homomorphism induced by the map $v \mapsto -v$ on V. α will occur frequently throughout topics using Clifford algebras. In particular, we can think of $Cl^+(V)$ and $Cl^-(V)$ as the eigenspaces of α.

Secondly, define the involution $^-$ by $e_I \mapsto e_I^3$. For greater convenience in computations, we note that for positive definite q and $|I| = k$, $e_I^3 = (-1)^{k(k+1)/2} e_I$ since $e_I e_I = (-1)^{k-1} e_{I'} e_{I'} e_{i_k} e_{i_k} = (-1)^k e_{I'} e_{I'} = \ldots = (-1)^{k(k+1)/2}$, where $I' = I \backslash \{i_k\}$. Although this map looks a bit awkward at first, it will seem quite familiar once we have seen the examples in the next section.

As these two involutions only depend on the degree of their argument, they are in fact canonical.

VOLUME ELEMENT Analogous to the exterior algebra, for an oriented orthonormal basis $\{e_1, \dots, e_n\}$ of $(V, q_{r,s})$, we define the *volume element* of $Cl_{r,s}$ to be $\omega = e_1 \dots e_n$.

Note that, for a given orientation, ω is independent of choice of oriented orthonormal basis: Consider another orthonormal basis of the same orientation $\{e'_1, \dots, e'_n\}$ with $e'_i = o_{ij} e_j$ where $o \in \mathrm{SO}_{r,s}$ and recall that the degree is invariant under change of orthonormal basis. Thus $\omega' = e'_1 \dots e'_n = o_{1j_1} \dots o_{nj_n} e_{j_1} \dots e_{j_n} = \sum_{\sigma \in \mathrm{S}_n} o_{1\sigma(1)} \dots o_{n\sigma(n)} \mathrm{sgn}(\sigma) e_1 \dots e_n = \det(o)\omega = \omega$.

It is easy to check that $\omega^2 = (-1)^{n(n+1)/2+s}$. Using our knowledge about the centre of the Clifford algebra and the involution α from above, we see that $\omega \in Z(Cl_{r,s})$ if $r+s$ is odd and $x\omega = \omega\alpha(x)$ otherwise.

Finally, we can define the *dual* map $\tilde{} : Cl^k \to Cl^{n-k}$ by $x \mapsto x\omega$. Note, that apart from the signs it resembles the Hodge star operator on forms.

5 FIRST EXAMPLES

Now we are set to compute a couple of basic examples. We let our field be the reals and use non-degenerate quadratic forms $q_{r,s}$.

$Cl(0)$ Since the Clifford algebra is defined to be an algebra with 1, this cannot be 0. $Cl(0) = \mathrm{span}_{\mathbf{R}}\{1\} \simeq \mathbf{R}$.

$Cl_{1,0}$ This Clifford algebra is generated by 1 and e_1 and we have the relation $e_1^2 = -q_{1,0}(e_1) = -1$. As an algebra this is isomorphic to $\mathbf{C}$ with $1 \mapsto 1$ and $e_1 \mapsto i$. Furthermore we have $Cl^0_{1,0} = Cl^+_{1,0} = \mathrm{span}_{\mathbf{R}}\{1\}$ and $Cl^1_{1,0} = Cl^-_{1,0} = \mathrm{span}_{\mathbf{R}}\{e_1\}$ and since n is odd, the centre of $Cl_{1,0}$ is all of it and thus, unsurprisingly, we see that $\mathbf{C}$ is commutative. The reversion $\bar{}$ defined in the previous section turns out to be complex conjugation as it maps 1 to 1 and i to $-i$.

$Cl_{0,1}$ While this Clifford algebra is similar to the previous as far as its structure as a vector space are concerned, the different quadratic form gives rise to a different algebra structure as we have $e_1^2 = -q_{0,1}(e_1) = 1$. Thus it is not isomorphic to $\mathbf{C}$ and we will denote it as $\mathbf{R} \oplus \mathbf{R}$. As an algebra it is isomorphic to $\mathbf{R}[X]/(X^2-1)$ but we are not really interested in this fact.

$Cl_{2,0}$ This Clifford algebra has the basis $\{1, e_1, e_2, e_1e_2\}$ and the relations $e_1^2 = e_2^2 = -1$, $(e_1e_2)^2 = -e_1^2e_2^2 = -1$ as well as $e_1e_2(e_1e_2) = -1$ hold. Thus $Cl_{2,0} \simeq \mathbf{H}$ via the algebra isomorphism given by $1 \mapsto 1$, $e_1 \mapsto i$, $e_2 \mapsto j$ and $e_1e_2 \mapsto k$.

Also, we see that the subalgebra $Cl_{2,0}^+ = \mathrm{span}_{\mathbf{R}}\{1, e_1e_2\} \simeq \mathbf{C}$. Now we can effortlessly see that $Z(\mathbf{H}) \simeq Cl_{2,0}^0 = \mathrm{span}_{\mathbf{R}}\{1\}$ as n is even, i.e. quaternionic multiplication is not commutative. Finally the reversion $^-$ turns out to be quaternionic conjugation and thus begins to feel familiar and natural as promised above.

$Cl_{1,1}$ Again, as a vector-space and with regard to its grading, this Clifford algebra is the same as the previous one. However, now we have the relations $e_1^2 = -1$, $e_2^2 = 1$ and $(e_1e_2)^2 = 1$. This is not isomorphic to the quaternion algebra. However we can find an algebra-isomorphism to the matrix algebra $\mathbf{R}(2)$ given by

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_2 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, e_1e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$Cl_{0,2}$ This example is very similar to the previous one. The Clifford algebra is isomorphic to $\mathbf{R}(2)$ via.

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_1 \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, e_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_1e_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

HIGHER DIMENSIONS Taking this approach of computing Clifford algebras to higher dimensions promises to be a lengthy and tedious task. Instead of doing this, we will find relations between the different $Cl_{r,s}$ that allow us to compute any Clifford algebra for a non-degenerate quadratic form over $\mathbf{R}$ from the basic examples we have already seen.

6 ISOMORPHISMS

We have the following isomorphisms:

$$\begin{aligned} Cl_{n,0} \otimes Cl_{0,2} &\simeq Cl_{0,n+2} \\ Cl_{0,n} \otimes Cl_{2,0} &\simeq Cl_{n+2,0} \\ Cl_{r,s} \otimes Cl_{1,1} &\simeq Cl_{r+1,s+1} \end{aligned}$$

To prove this, we give the relevant isomorphisms. Consider $(\mathbf{R}^{n+2}, q_{0,n+2})$ with basis $\{\varepsilon_1, \dots, \varepsilon_{n+2}\}$, $Cl_{n,0}$ generated by $\{f_1, \dots, f_n\}$ and $Cl_{0,2}$ generated by $\{\gamma_1, \gamma_2\}$. Then define the linear map

$$\begin{aligned} \varphi : (\mathbf{R}^{n+2}, q_{0,n+2}) &\longrightarrow Cl_{n,0} \otimes Cl_{0,2} \\ \varepsilon_i &\longmapsto \begin{cases} f_i \otimes \gamma_1\gamma_2 & i \leq n \\ 1 \otimes \gamma_{i-n} & i > n \end{cases} . \end{aligned}$$

Now we can compute for $i, j \leq n$ $\varphi(\varepsilon_i)\varphi(\varepsilon_j) = f_i f_j \otimes \gamma_1\gamma_2\gamma_1\gamma_2 = f_i f_j \otimes (-1)$. In particular, for $i \neq j$ we have $\varphi(\varepsilon_i)\varphi(\varepsilon_j) = -\varphi(\varepsilon_j)\varphi(\varepsilon_i)$. Similarly, for $n < k, l$, $\varphi(\varepsilon_k)\varphi(\varepsilon_l) = 1 \otimes \gamma_k\gamma_l$ which again for $k \neq l$ implies that $\varphi(\varepsilon_k)\varphi(\varepsilon_l) = -\varphi(\varepsilon_l)\varphi(\varepsilon_k)$. Finally, for $i \leq n < k$, we have $\varphi(\varepsilon_i)\varphi(\varepsilon_k) = f_i \otimes \gamma_1\gamma_2\gamma_{k-n} = -\varphi(\varepsilon_k)\varphi(\varepsilon_i)$. Use the antisymmetry of all the preceding equalities to see that for $x = x_i e_i$, we have

$$\varphi(x)\varphi(x) = x_i x_j \varphi(\varepsilon_i)\varphi(\varepsilon_j) = x_i x_j \delta_{ij} 1 \otimes 1 = x_i^2 1 \otimes 1 = -q_{0,n+2}(x) 1 \otimes 1$$

since all terms with $i \neq j$ cancel. This, however, is just the condition $(*)$ and thus, by the universal property, φ descends to an algebra-homomorphism $\tilde{\varphi} : Cl_{0,n+2} \to Cl_{n,0} \otimes Cl_{0,2}$. As $\tilde{\varphi}$ is surjective and both vector spaces have the same dimension, $\tilde{\varphi}$ is an isomorphism. The map we need to prove the second isomorphism is analogous to the one used here and the proof runs along the same lines.

For the third isomorphism, the strategy is exactly the same, only the map is more elaborate: With $\{e_1, \ldots, e_{r+1}, \varepsilon_1, \ldots, \varepsilon_{s+1}\}$ a basis for $\mathbf{R}^{(r+1)+(s+1)}$, $\{f_1, \ldots, f_r, \eta_1, \ldots, \eta_s\}$ generators for $Cl_{r,s}$ and $\{g_1, \gamma_1\}$ generators for $Cl_{1,1}$, we define a linear map

$$\begin{array}{rcl} \varphi : (\mathbf{R}^{(r+1)+(s+1)}, q_{r+1,s+1}) & \longrightarrow & Cl_{r,s} \otimes Cl_{1,1} \\ e_i & \longmapsto & \begin{cases} f_i \otimes g_1\gamma_1 & i \leq r \\ 1 \otimes g_1 & i = r+1 \end{cases} \\ \varepsilon_i & \longmapsto & \begin{cases} \eta_i \otimes g_1\gamma_1 & i \leq s \\ 1 \otimes \gamma_1 & i = s+1 \end{cases} . \end{array}$$

Doing a couple of easy but annoying-to-type computations gives that we have the same antisymmetry relations as above and thus we get for $x = x_i e_i + \xi_j \varepsilon_j$:

$$\begin{array}{rl} & \varphi(x)^2 \\ = & \varphi(x_i e_i)\varphi(x_k e_k) + \overbrace{\varphi(x_i e_i)\varphi(\xi_l \varepsilon_l) + \varphi(\xi_j \varepsilon_j)\varphi(x_k e_k)}^{=0} + \varphi(\xi_j \varepsilon_j)\varphi(\xi_l \varepsilon_l) \\ = & -x_i x_k \delta_{ik} 1 \otimes 1 + \xi_j \xi_l \delta_{jl} 1 \otimes 1 \\ = & -q_{r,s}(x) 1 \otimes 1. \end{array}$$

Thus this map also satisfies equation $(*)$ and descends to a map on the Clifford algebra, giving the desired isomorphism.

We can in principle compute all of the $Cl_{r,s}$ using the Clifford algebras discussed in the examples and the isomorphisms given above. But life is even easier as we have the following 'periodic' isomorphisms that use Clifford algebras of positive or negative definite quadratic forms only, where the previous isomorphisms mixed both of them:

$$\begin{array}{rcl} Cl_{n,0} \otimes Cl_{8,0} & \simeq & Cl_{n+8,0} \\ Cl_{0,n} \otimes Cl_{0,8} & \simeq & Cl_{0,n+8} \end{array}$$

Proving that these isomorphisms exist is not difficult when assuming the knowledge of the isomorphism $\mathbf{H} \otimes \mathbf{H} \simeq \mathbf{R}(4)$ proved in the appendix, the examples of section 5 and repeatedly applying the isomorphisms we have just seen. We have $Cl_{n+8,0} \simeq Cl_{n,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{2,0} \simeq Cl_{n,0} \otimes \mathbf{R}(2) \otimes \mathbf{H} \otimes \mathbf{R}(2) \otimes \mathbf{H} \simeq Cl_{n,0} \otimes \mathbf{R}(4) \otimes \mathbf{R}(4) \simeq Cl_{n,0} \otimes \mathbf{R}(16)$ where $\mathbf{R}(16) \simeq \mathbf{R}(16) \otimes \mathbf{R} \simeq Cl_{8,0}$. Exactly the same argument works for the second isomorphism and in particular we have $Cl_{8,0} \simeq \mathbf{R}(16) \simeq Cl_{0,8}$.

As can be seen from the proof, we could as well establish similar isomorphisms with a periodicity of 4 only. However, the given isomorphisms are preferred as they allow us to think of $Cl_{n+8,0}$ and $Cl_{0,n+8}$ as the algebras of 16×16 matrices with entries in $Cl_{n,0}$ and $Cl_{0,n}$ respectively.

7 FURTHER EXAMPLES

Given the first three isomorphisms of the previous section, we can compute further Clifford algebras. To do this we first note the following basic isomorphisms of tensor products:

$$\mathbf{R}(n) \otimes \mathbf{R}(m) \simeq \mathbf{R}(nm) \qquad \mathbf{R}(n) \otimes \mathbf{C} \simeq \mathbf{C}(n) \qquad \mathbf{R}(n) \otimes \mathbf{H} \simeq \mathbf{H}(n)$$

We will also need the following less obvious isomorphisms that are proved in the appendix:

$$\mathbf{C} \otimes \mathbf{H} \simeq \mathbf{C}(2) \qquad \mathbf{H} \otimes \mathbf{H} \simeq \mathbf{R}(4)$$

Starting with the six Clifford algebras we have already computed in section 5, we can now compute all the others.

DIMENSION 3 Unfortunately we cannot expect to have octonions here, as we are considering associative algebras only.

$$\begin{aligned}
Cl_{3,0} &\simeq Cl_{0,1} \otimes Cl_{2,0} \simeq (\mathbf{R} \oplus \mathbf{R}) \otimes \mathbf{H} \simeq \mathbf{H} \oplus \mathbf{H} \\
Cl_{2,1} &\simeq Cl_{1,0} \otimes Cl_{1,1} \simeq \mathbf{C} \otimes \mathbf{R}(2) \simeq \mathbf{C}(2) \\
Cl_{1,2} &\simeq Cl_{0,1} \otimes Cl_{1,1} \simeq (\mathbf{R} \oplus \mathbf{R}) \otimes \mathbf{R}(2) \simeq \mathbf{R}(2) \oplus \mathbf{R}(2) \\
Cl_{0,3} &\simeq Cl_{1,0} \otimes Cl_{0,2} \simeq \mathbf{C} \otimes \mathbf{R}(2) \simeq \mathbf{C}(2).
\end{aligned}$$

DIMENSION 4

$$\begin{aligned}
Cl_{4,0} &\simeq Cl_{0,2} \otimes Cl_{2,0} \simeq \mathbf{R}(2) \otimes \mathbf{H} \simeq \mathbf{H}(2) \\
Cl_{3,1} &\simeq Cl_{2,0} \otimes Cl_{1,1} \simeq \mathbf{H} \otimes \mathbf{R}(2) \simeq \mathbf{H}(2) \\
Cl_{2,2} &\simeq Cl_{1,1} \otimes Cl_{1,1} \simeq \mathbf{R}(2) \otimes \mathbf{R}(2) \simeq \mathbf{R}(4) \\
Cl_{1,3} &\simeq Cl_{0,2} \otimes Cl_{1,1} \simeq \mathbf{R}(2) \otimes \mathbf{R}(2) \simeq \mathbf{R}(4) \\
Cl_{0,4} &\simeq Cl_{2,0} \otimes Cl_{0,2} \simeq \mathbf{H} \otimes \mathbf{R}(2) \simeq \mathbf{H}(2)
\end{aligned}$$

DIMENSION 5

$$\begin{aligned}
Cl_{5,0} &\simeq Cl_{0,3}\otimes Cl_{2,0}\simeq \mathbf{C}(2)\otimes\mathbf{H}\simeq\mathbf{R}(2)\otimes\mathbf{C}\otimes\mathbf{H}\simeq\mathbf{C}(4)\\
Cl_{4,1} &\simeq Cl_{3,0}\otimes Cl_{1,1}\simeq (\mathbf{H}\oplus\mathbf{H})\otimes\mathbf{R}(2)\simeq\mathbf{H}(2)\oplus\mathbf{H}(2)\\
Cl_{3,2} &\simeq Cl_{2,1}\otimes Cl_{1,1}\simeq \mathbf{C}(2)\otimes\mathbf{R}(2)\simeq\mathbf{C}(4)\\
Cl_{2,3} &\simeq Cl_{1,2}\otimes Cl_{1,1}\simeq (\mathbf{R}(2)\oplus\mathbf{R}(2))\otimes\mathbf{R}(2)\simeq\mathbf{R}(4)\oplus\mathbf{R}(4)\\
Cl_{1,4} &\simeq Cl_{0,3}\otimes Cl_{1,1}\simeq \mathbf{C}(2)\otimes\mathbf{R}(2)\simeq\mathbf{C}(4)\\
Cl_{0,5} &\simeq Cl_{3,0}\otimes Cl_{0,2}\simeq (\mathbf{H}\oplus\mathbf{H})\otimes\mathbf{R}(2)\simeq\mathbf{H}(2)\oplus\mathbf{H}(2)
\end{aligned}$$

FURTHER CASES We can go on computing Clifford algebras like this until we become very bored. As Clifford algebras of spaces with positive definite quadratic forms are the most important for us, for the time being we shall just compute

$$\begin{aligned}
Cl_{6,0} &\simeq Cl_{0,4}\otimes Cl_{0,2}\simeq\mathbf{H}(2)\otimes\mathbf{H}\simeq\mathbf{R}(2)\otimes\mathbf{H}\otimes\mathbf{H}\simeq\mathbf{R}(2)\otimes\mathbf{R}(4)\\
&\simeq \mathbf{R}(8)\\
Cl_{7,0} &\simeq Cl_{0,5}\otimes Cl_{0,2}\simeq(\mathbf{H}(2)\oplus\mathbf{H}(2))\otimes\mathbf{H}\simeq(\mathbf{H}(2)\otimes\mathbf{H})\oplus(\mathbf{H}(2)\otimes\mathbf{H})\\
&\simeq \mathbf{R}(8)\oplus\mathbf{R}(8)\\
Cl_{8,0} &\simeq \mathbf{R}(16)\quad\text{as seen before.}
\end{aligned}$$

Having in mind the periodic isomorphism we have seen above, for $n > 8$ with $n = 8a + b$ we have

$$Cl_{n,0}\simeq Cl_{b,0}\otimes\bigotimes^{a}\mathbf{R}(16)\simeq Cl_{b,0}\otimes\mathbf{R}(16^{a})$$

SUMMARY To summarise our computations, I give a table of the frequently used Clifford algebras $Cl_{n,0}$. For a comprehensive table of the $Cl_{r,s}$ for $r, s \leq 8$ see [1, p. 29].

n	0	1	2	3	4	5	6	7	8
$Cl_{n,0}$	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H}\oplus\mathbf{H}$	$\mathbf{H}(2)$	$\mathbf{C}(4)$	$\mathbf{R}(8)$	$\mathbf{R}(8)\oplus\mathbf{R}(8)$	$\mathbf{R}(16)$

8 MORE ISOMORPHISMS

We have seen the isomorphisms we need to compute all $Cl_{r,s}$. But there exist more isomorphims revealing further parts of the relations between Clifford algebras and their structures. There are more isomorphisms similar to those we have seen in section 6, revealing for example that the periodicity proved exists also for spaces with indefinite quadratic forms and that the table of $Cl_{r,s}$ just mentioned will be symmetric with respect to the line $s = r+1$. The proofs of these statements run exactly along the same lines as the argument given in the beginning of section 6. Rather than going through the computations to do this, we will see a couple of different and new isomorphisms.

EVEN PARTS $Cl_{r,s}$ is isomorphic as an algebra to the even part of $Cl_{r+1,s}$ via the following isomorphism: Consider $(\mathbf{R}^{(r+1)+s}, q_{r+1,s})$ and its subspace $(R^{r+s}, q_{r,s}) = (\mathrm{span}_{\mathbf{R}}\{e_i | i \neq r+1\}, q_{r,s})$ and define the linear map

$$\begin{aligned} \varphi : (R^{r+s}, q_{r,s}) &\longrightarrow Cl^+_{r+1,s} \\ e_i &\longmapsto e_{r+1}e_i. \end{aligned}$$

Note that φ is well-defined. Then, with $v = \sum_{i \neq r+1} v_i e_i$, we have

$$\varphi(v)^2 = \sum_{i,j \neq r+1} v_i v_j e_{r+1} e_i e_{r+1} e_j = \sum_{i,j \neq r+1} v_i v_j e_i e_j = v^2 = -q_{r,s} 1$$

Thus, φ satisfies $(*)$ and by the universal property it can be extended to an algebra-homomorphism $\tilde{\varphi} : Cl_{r,s} \to Cl^+_{r+1,s}$. As $\tilde{\varphi}(e_I)$ is $\pm e_I$ or $\pm e_{r+1} e_I$, both of which being non-zero, we have that $\tilde{\varphi}$ is injective. Comparing dimensions shows that $\tilde{\varphi}$ is an isomorphism as desired.

Note, that for $Cl_{1,0}$ and $Cl_{2,0}$ this gives us copies of $\mathbf{R}$ and $\mathbf{C}$ that we expect to find in $\mathbf{C}$ and $\mathbf{H}$ respectively.

COMPLEX CASE Let us assume $K = \mathbf{C}$ for the moment. Recall that the complex version of Sylvester's theorem tells us that, since for $q(e_j) = -1$ we have $q(ie_j) = 1$, we can choose our basis such that all non-degenerate quadratic forms are of the form $q_{n,0}$. In particular, this quadratic form is the complexification $q_{r,s} \otimes 1_{\mathbf{C}} : \mathbf{R}^{r+s} \otimes \mathbf{C} \to \mathbf{R} \otimes \mathbf{C} \simeq \mathbf{C}$ for any $r + s = n$. Thus, $Cl(\mathbf{C}^n, q_{\mathbf{C}}) \simeq Cl(\mathbf{R}^{r+s} \otimes \mathbf{C}, q_{r,s} \otimes 1_{\mathbf{C}}) \simeq Cl_{r,s} \otimes \mathbf{C}$ for all $r + s = n$. This is the *complex Clifford algebra* which we will denote by $\mathbf{C}l_n$. Basic examples are:

$$\begin{aligned} \mathbf{C}l_0 &= \mathbf{C} \\ \mathbf{C}l_1 &= Cl_{1,0} \otimes \mathbf{C} = \mathbf{C} \otimes \mathbf{C} \simeq \mathbf{C} \oplus \mathbf{C} \\ \mathbf{C}l_2 &= Cl_{0,2} \otimes \mathbf{C} = \mathbf{R}(2) \otimes \mathbf{C} \simeq \mathbf{C}(2). \end{aligned}$$

Again, this is all we need to know, as we have a very short 'periodicity' isomorphism

$$\mathbf{C}l_{n+2} = Cl_{n+2,0} \otimes \mathbf{C} \simeq (Cl_{0,n} \otimes Cl_{2,0}) \otimes \mathbf{C} \simeq \mathbf{C}l_n \otimes_{\mathbf{C}} \mathbf{C}l_2$$

Note, that in the last step we need the complex tensor product as otherwise we wouldn't have a complex vector space.

DIRECT SUMS If (V, q) is an orthogonal direct sum $(V_1, q_1) \oplus (V_2, q_2)$, i.e. we can write elements v of V uniquely as $v_1 + v_2$ with $v_i \in V_i$ and $q(v_1 + v_2) = q(v_1) + q(v_2)$, we have a linear map

$$\begin{aligned} \varphi : (V, q) &\longrightarrow Cl(V_1, q_1) \otimes^{\mathbf{Z}_2} Cl(V_2, q_2) \\ v_1 + v_2 &\longmapsto v_1 \otimes 1 + 1 \otimes v_2. \end{aligned}$$

where $\otimes^{\mathbf{Z}_2}$ denotes the $\mathbf{Z}_2$-graded tensor product with $\mathbf{Z}_2$-graded multiplication $(v_1 \otimes v_2)(w_1 \otimes w_2) := (-1)^{\deg(v_2)\deg(w_1)} v_1 w_1 \otimes v_2 w_2$ of the $\mathbf{Z}_2$ graded Clifford algebras. We observe that $\varphi(v)^2 = \varphi(v_1 \otimes 1 + 1 \otimes v_2)^2 = v_1^2 \otimes 1 + v_1 \otimes v_2 - v_1 \otimes v_2 + 1 \otimes v_2^2 = -(q(v_1) + q(v_2))1 \otimes 1 = -q(v)1 \otimes 1$ and thus φ satisfies $(*)$ and by the universal property extends to an algebra homomorphism $\tilde{\varphi} : Cl(V, q) \to Cl(V_1, q_1) \otimes^{\mathbf{Z}_2} Cl(V_2, q_2)$ that is surjective and thus an isomorphism by the usual argument.

In particular, having in mind the decomposition of $(\mathbf{R}^{r+s}, q_{r,s})$ into the direct sum $(\mathrm{span}_{\mathbf{R}}\{e_1\}, q_{1,0}) \oplus \ldots \oplus (\mathrm{span}_{\mathbf{R}}\{e_r\}, q_{1,0}) \oplus (\mathrm{span}_{\mathbf{R}}\{e_{r+1}\}, q_{0,1}) \oplus \ldots \oplus (\mathrm{span}_{\mathbf{R}}\{e_{r+s}\}, q_{0,1})$ and using the isomorphism repeatedly gives us

$$Cl_{r,s} \simeq \bigotimes_{r \text{ times}}^{\mathbf{Z}_2} Cl_{1,0} \quad \otimes^{\mathbf{Z}_2} \bigotimes_{s \text{ times}}^{\mathbf{Z}_2} Cl_{0,1}.$$

Although this shows that we don't really need the isomorphisms we have discussed before to compute all of the $Cl_{r,s}$, the method we just discovered is not very practical.

SPLITTING Recall the definition and properties of the volume element ω in section 4. Then, for odd n, ω commutes with all other elements and so do the elements $\eta^{\oplus} = \frac{1}{2}(1 + \omega)$ and $\eta^{\ominus} = \frac{1}{2}(1 - \omega) = \alpha(\eta^{\oplus})$ which we use to define the two-sided ideals $Cl_{r,s}^{\oplus} = \eta^{\oplus} Cl_{r,s}$ and $Cl_{r,s}^{\ominus} = \eta^{\ominus} Cl_{r,s}$.

Assuming that $\omega^2 = 1$, we have that firstly $\eta^{\oplus} + \eta^{\ominus} = 1$, implying $Cl_{r,s}^{\oplus} + Cl_{r,s}^{\ominus} = Cl_{r,s}$, secondly $\eta^{\oplus}\eta^{\ominus} = 0 = \eta^{\ominus}\eta^{\oplus}$, implying that $Cl_{r,s}^{\oplus} \cap Cl_{r,s}^{\ominus} = 0$, and thirdly both $\eta^{\oplus}$ and $\eta^{\ominus}$ are idempotent, giving $Cl_{r,s}^{\oplus} Cl_{r,s}^{\oplus} \subset Cl_{r,s}^{\oplus}$ and $Cl_{r,s}^{\ominus} Cl_{r,s}^{\ominus} \subset Cl_{r,s}^{\ominus}$.

Together, these three facts give that $Cl_{r,s}$ splits into a direct sum of algebras $Cl_{r,s}^{\oplus} \oplus Cl_{r,s}^{\ominus}$. Furthermore, since $\eta^{\oplus}$ and $\eta^{\ominus}$ are interchanged by α, so are $Cl_{r,s}^{\oplus}$ and $Cl_{r,s}^{\ominus}$. As α is an involution, these two subalgebras are isomorphic.

The fact we have just proved, tells us which $Cl_{r,s}$ we can expect to be of the curious form $V \oplus V$ that we have encountered when computing the examples earlier. We have seen that this will be the case for $r + s$ odd and $\omega^2 = 1$.

9 FINAL WOFFLE

The treatment I have given seems to be pretty much standard. In fact, most of the textbooks follow closely the exposition originally given in [3]. For our purposes, next we want to introduce the Spin-group constructed from Clifford algebras and see that it is the simply connected double cover of the special orthogonal group (cf. [4] or [1, I,§2]) and then proceed to define Spin structures on vector bundles, Clifford bundles (cf. [6] or [1, II]) and finally proceed to the definition and properties of the Dirac operator (cf. [4] or [1, II]).

However, Clifford algebras are also interesting in their own right. At a closer look Clifford multiplication can be decomposed into an antisymmetric and a symmetric part $vw = v \wedge w - \langle v, w\rangle$ where $\wedge$ and $\langle \cdot, \cdot \rangle$ restrict to the outer and inner product on V respectively and thus Clifford multiplication relates the inner and the outer product. This, for example sheds light on the curious relation between the wedge product and the cross product on $\mathbf{R}^3$. We can see that for $v, w \in \mathbf{R}^3$ we have $v \times w = \widetilde{v \wedge w} = (v \wedge w)\omega$.

Apart from such algebraic niceties, the Clifford algebra can also be given a geometric interpretation. For example we can model projective geometry by representing a point in $\mathbf{RP}^{n-1}$ by a non-zero vector in $\mathbf{R}^n$ as usual. Then, for distinct points p and q, the line they determine can be represented by the element $p \wedge q$ in the Clifford algebra. Similarly the *meet* of two projective lines L and M will be given in terms of $\wedge$ and $\tilde{}$. In particular the *duality in projective geometry* will be given by $\tilde{}$.

Having these tools at our disposal, we have an algebraic way to state and prove the classical theorems of projective geometry such as Desargues' and Pappus-Pascal's by purely algebraic means in a co-ordinate free way and without need to use homogeneous co-ordinates or worry about 'points at infinity'. For more on this you can refer to [5] and the items of its bibliography.

APPENDIX AUXILIARY ISOMORPHISMS

When establishing the 'periodicity' isomorphism on page 7 and computing the examples in section 7 we used without proof the isomorphisms

$$\mathbf{C} \otimes \mathbf{H} \simeq \mathbf{C}(2) \qquad \text{and} \qquad \mathbf{H} \otimes \mathbf{H} \simeq \mathbf{R}(4).$$

To prove the first one, think of $\mathbf{H}$ as a (left) $\mathbf{C}$-module to define the map

$$\varphi : \mathbf{C} \times \mathbf{H} \longrightarrow \mathrm{Hom}_{\mathbf{C}}(\mathbf{H}, \mathbf{H}) \qquad (z, q) \longmapsto (\varphi_{z,q} : x \mapsto zx\bar{q}) .$$

Note that φ respects the algebra structure on $\mathbf{C} \times \mathbf{H}$ as

$$\varphi_{z,q}\varphi_{z',q'}(x) = \varphi_{z,q}(z'x\bar{q'}) = zz'x\bar{q'}\bar{q} = (zz')x\overline{(qq')} = \varphi_{zz',qq'}(x)$$

and also φ is $\mathbf{R}$-bilinear thus, by the universal property of the tensor product, descends to an algebra-homomorphism $\tilde{\varphi} : \mathbf{C} \otimes \mathbf{H} \to \mathrm{Hom}_{\mathbf{C}}(\mathbf{H}, \mathbf{H})$. Since $\tilde{\varphi}_{z\otimes q} = 0$ if and only if $z \otimes q = 0$, we see that $\tilde{\varphi}$ is injective and as $\dim_{\mathbf{R}} \mathbf{C} \otimes \mathbf{H} = 8 = \dim_{\mathbf{R}} \mathrm{Hom}_{\mathbf{C}}(\mathbf{H}, \mathbf{H})$ it is in fact an isomorphism. Noting that $\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}, \mathbf{H}) \simeq \mathbf{C}(2)$ completes the proof of the first isomorphism.

The proof for the second isomorphism runs along the same lines: Define

$$\varphi : \mathbf{H} \times \mathbf{H} \longrightarrow \mathrm{Hom}_{\mathbf{R}}(\mathbf{H}, \mathbf{H}) \qquad (q, p) \longmapsto (\varphi_{q,p} : x \mapsto qx\bar{p}) .$$

By similar arguments to those above, we see that φ is compatible with the algebra-structure as well as $\mathbf{R}$-bilinear and thus descends to an algebra-homomorphism $\tilde{\varphi} : \mathbf{H} \otimes \mathbf{H} \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{H}, \mathbf{H})$ that is injective and thus for reasons of dimension an isomorphism. Again, noting that $\mathrm{Hom}_{\mathbf{R}}(\mathbf{H}, \mathbf{H}) \simeq \mathbf{R}(4)$ completes the proof.

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