SYMPLECTIC AND SPECIAL LAGRANGIAN GEOMETRY

M.SC. DISSERTATION*

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ABSTRACT

This text gives an introduction to symplectic geometry, proceeding to moment maps and symplectic reduction, which is illustrated with examples. The notions of Lagrangian and special Lagrangian subspaces and submanifolds are introduced and discussed. An example for a special Lagrangian fibration is given.

INTRODUCTION

In this text some of the technical prerequisites for approaching the topic of mirror symmetry are presented. In [10, §11] mirror symmetry is briefly described as "a mysterious relationship between pairs of Calabi-Yau 3-folds [. . .] arising from a branch of physics known as String Theory, and leading to some very strange and exciting conjectures about Calabi-Yau 3-folds [. . .]" – and thus is beyond the scope of this dissertation. A sound knowledge of manifolds along with some knowledge of Lie groups and linear algebra should suffice to understand this text.

The first topic treated is symplectic geometry which is a huge topic in its own right. The theory is developed to facilitate the understanding of moment maps and symplectic reduction. Intriguing topics such as 'nonsqueezing' properties [14, §§2.4, 12.1] or the question of the existence of symplectic structures have been left out and most paragraphs in the text could be extended to whole sections with further facts and examples.

Examples are given from the definition of a symplectic manifold and developed all the way to the process of symplectic reduction. The examples from classical mechanics are basic – owing to the author's ignorance of physics.

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For symplectic geometry Berndt's book [2] is useful introductory reading and often a subset of Abraham and Marsden's detailed and comprehensive book on mechanics [1] which is rightly shelved along with the mathematics books in the library. An even broader view with less computations is given in [14]. Originally, symplectic reduction, which is described in all of the books mentioned, was introduced by Marsden and Weinstein in [13].

Harvey and Lawson's paper on calibrated geometries [9] was the foundational work on that topic and gives a good introduction to Lagrangian and special Lagrangian subspaces and submanifolds. Using this and the setting of almost Calabi-Yau manifolds, a few facts from Gross [7] and Goldstein [5] are established and illustrated by an example for a special Lagrangian fibration.

The next step would be to further investigate special Lagrangian fibrations, allow them to have singularities and study these singularities and their behaviour – using techniques from both algebraic and differential geometry – see [7] or [10, §10]. Eventually, it is hoped that this helps in understanding mirror symmetry, the SYZ-conjecture and their relation to string theory in physics – see [10, §11].

As there are still many open questions in this area of mathematics, it will be an interesting an exciting field for further study.

1 SYMPLECTIC VECTOR SPACES

DEFINITION To begin with, we need to familiarise ourselves with the notion of a *symplectic vector space* and the basic properties of such spaces: A *symplectic vector space* is a pair (V, ω), where V is a finite dimensional $\mathbf{R}$-vector space and $\omega : V \times V \to \mathbf{R}$ is a *(linear) symplectic form*, i.e. a skew-symmetric non-degenerate bilinear form.

We can think of ω as an element of $\Lambda^2 V^*$. Furthermore, by non-degeneracy, ω gives rise to an isomorphism $\omega^\flat : V \to V^*$ given by $v \mapsto \iota_v \omega$, i.e. by the contraction of ω with v. In fact $\omega^\flat$ being an isomorphism is equivalent to non-degeneracy of ω, as is – in the case of $2n$-dimensional spaces – the condition that ω^n is non-zero.

SUBSPACES Given a symplectic vector space (V, ω) and a subspace $W \subset V$, we can define the *ω-orthogonal complement of W in V*:

$$W^\omega = \{v \in V \mid \forall w \in W : \omega(v, w) = 0\}.$$

Using this definition twice we see that $W^{\omega\omega} = W$ and using $\omega^\flat$ to identify W^ω with those elements of V^* annihilating all of W, we get the dimension formula $\dim W^\omega + \dim W = \dim V$.

However, the intersection of W and W^ω does not need to be trivial. If it is, W is called a *symplectic* subspace and the restriction of ω to W is non-degenerate . Using that $W^{\omega\omega} = W$, it is immediate that if W is a symplectic subspace, so is W^ω.

There are special names for the other cases as well: W is called *isotropic* if $W \subset W^\omega$, *coisotropic* if $W^\omega \subset W$ and *Lagrangian* if it is both isotropic and coisotropic, i.e. if $W = W^\omega$. By the dimension formula we see that Lagrangian subspaces must be of dimension $\dim V/2$.

BASIS The first fact we arrive at is that for any symplectic vector space (V, ω) we can find a basis $\{q_1, p_1, \dots, q_n, p_n\}$ such that $\omega(q_j, p_k) = \delta_{jk}$ and $\omega(q_j, q_k) = \omega(p_j, p_k) = 0$ for $1 \leq j, k \leq n$. A basis with these properties is called a *symplectic basis.*

To prove this, choose a non-zero vector q_1. Then, by non-degeneracy of ω, there is a vector p_1 such that $\omega(q_1, p_1) = 1$. Let V_1 be the subspace spanned by those two vectors. Then V_1 is a symplectic subspace and thus, by the results of the previous paragraph, we have $V = V_1 \oplus V_1^\omega$. As V_1^ω is again symplectic, we can proceed inductively until we have the desired basis.

By this construction a symplectic basis has an even number of elements and thus all symplectic vector spaces are of even dimension. Symplectic bases are the symplectic analogue to the orthonormal bases we construct for inner product spaces by the Gram-Schmidt process.

EXAMPLE Now we are in a position to comfortably construct an example: Let $V = \mathbf{R}^{2n}$ and $\{q_1, p_1, \dots, q_n, p_n\}$ a basis of V. Then, with

$$v = \sum_{j=1}^{n} (a_j q_j + b_j p_j) \quad \text{and} \quad v' = \sum_{j=1}^{n} (a'_j q_j + b'_j p_j),$$

define $\omega(v, v') = \sum_{j=1}^{n} (a_j b'_j - a'_j b_j)$. This makes (V, ω) a symplectic vector space and $\{q_1, p_1, \dots, q_n, p_n\}$ a symplectic basis. The symplectic form ω we defined is called the *standard symplectic form for* $\mathbf{R}^{2n}$.

For a more concrete example, consider $\mathbf{C}^2$ as a $\mathbf{R}$-vector space with elements (z_1, z_2) where $z_j = x_j + iy_j$. Then

$$\omega((z_1, z_2), (z'_1, z'_2)) = x_1 y'_1 + x_2 y'_2 - x'_1 y_1 - x'_2 y_2$$

makes (V, ω) a symplectic vector space and $\{(1, 0), (i, 0), (0, 1), (0, i)\}$ is a symplectic basis. Furthermore, we can look at different subspaces and their ω-orthogonal complements and check whether they have any of the properties defined in the paragraph on subspaces:

W	W^ω	$W \cap W^\omega$	property
$\mathbf{R} \times 0$	$\mathbf{R} \times \mathbf{C}$	$\mathbf{R} \times 0$	isotropic
$\mathbf{C} \times 0$	$0 \times \mathbf{C}$	0	symplectic
$\mathbf{R} \times \mathbf{R}$	$\mathbf{R} \times \mathbf{R}$	$\mathbf{R} \times \mathbf{R}$	Lagrangian
$\mathbf{R} \times i\mathbf{R}$	$\mathbf{R} \times i\mathbf{R}$	$\mathbf{R} \times i\mathbf{R}$	Lagrangian
$\mathbf{C} \times \mathbf{R}$	$0 \times \mathbf{R}$	$0 \times \mathbf{R}$	coisotropic

MORPHISMS Given symplectic vector spaces (V, ω) and (V', ω'), we consider linear maps $L : V \to V'$ preserving the symplectic form, i.e. such that $L^*\omega' = \omega$, meaning that $\omega'(Lv, Lw) = \omega(v, w)$ for all $v, w \in V$. Since L preserves the nondegenerate form ω, it has to be an isomorphism. Such linear maps are called *(linear) symplectomorphisms*. The symplectomorphisms of (V, ω) to itself form the *symplectic (linear) group for* (V, ω): $\mathrm{Sp}(V, \omega)$.

CLASSIFICATION It turns out that any $2n$-dimensional symplectic vector space (V, ω') is symplectomorphic to $(\mathbf{R}^{2n}, \omega)$ where ω is the standard symplectic form: Let $\{q_1, p_1, \dots q_n, p_n\}$ and $\{q'_1, p'_1, \dots, q'_n, p'_n\}$ be symplectic bases for V and $\mathbf{R}^{2n}$ respectively, then

$$L : V \to \mathbf{R}^{2n} \qquad q_j \mapsto q'_j \quad p_j \mapsto p'_j$$

satisfies $L^*\omega = \omega'$ and thus is a symplectomorphism as desired. Due to the fact we have just proved there is no need for further examples of symplectic vector spaces. Also, all the symplectic groups are isomorphic to $\mathrm{Sp}(\mathbf{R}^{2n}, \omega)$ by conjugation and we write $\mathrm{Sp}(n)$ as as shorthand.

COMPLEX STRUCTURES It is interesting to see how symplectic forms ω relate to complex structures J, inner products g and hermitian forms h, i.e. automorphisms with the property $J^2 = -\mathrm{id}$, non-degenerate positive-definite symmetric bilinear forms and non-degenerate positive-definite complex-conjugate-skew-symmetric $\mathbf{C}$-bilinear forms respectively.

This can be approached in different ways, most of which involve choosing canonical bases for the respective structures and manipulating matrices to see how they are related and what they look like when the structures involved exist and are compatible. See, for example, [3, §2] for the computations leading to the relations

$$g(v, w) = \omega(v, Jw) \qquad \text{and} \qquad h(v, w) = g(v, w) + i\omega(v, w) \tag{A}$$

the latter of which requires us to identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ as needed. Looking at these relations in terms of the structure-preserving groups involved, we have

structure	ω	g	J	h
group	$\mathrm{Sp}(n)$	$\mathrm{O}(2n)$	$\mathrm{Gl}(n, \mathbf{C})$	$\mathrm{U}(n)$

As the h can be recovered from any two of ω, g, J and vice versa, this gives the following fact on the structure-preserving groups:

$$\mathrm{Sp}(n) \cap \mathrm{Gl}(n, \mathbf{C}) \;=\; \mathrm{Sp}(n) \cap \mathrm{O}(2n) \;=\; \mathrm{O}(2n) \cap \mathrm{Gl}(n, \mathbf{C}) \;=\; \mathrm{U}(n).$$

DEFINITION Now we transfer the notion of a symplectic form on a vector space to manifolds. Let M be a (smooth) manifold, then a *symplectic structure on* M is a closed non-degenerate 2-form $\omega \in \Omega^2(M)$. Thus, at any point $m \in M$, the tangent space is a symplectic vector space $(T_m M, \omega_m)$. A pair (M, ω) is called a *symplectic manifold*.

This definition is similar to that of a Riemannian manifold, where our additional structure, the Riemannian metric, is a smooth section of non-degenerate elements of the bundle S^2M, thus making the tangent space at any point an inner product space. However, the definition of a symplectic structure is more restrictive as it does not only consist of the algebraic condition that the form be non-degenerate on each tangent space but also adds the analytic condition $d\omega = 0$.

Given two symplectic manifolds (M, ω) and (M', ω') and a smooth map $f : M \to M'$ then this map is called a *symplectomorphism* if it preserves the symplectic structure, i.e. if $\omega = f^*\omega'$. That is, if

$$\omega_m(v, w) = \omega'_{f(m)}(d_m f(v), d_m f(w)) \quad \text{for all } m \in M \text{ and } v, w \in T_m M.$$

The set of all symplectomorphisms from a symplectic manifold (M, ω) to itself is denoted $\mathrm{Sp}(M, \omega)$.

FACTS There are a few facts about symplectic manifolds which are worth mentioning and easy to establish. Firstly, symplectic manifolds have to be of even dimension, as their tangent spaces are symplectic vector spaces and thus of even dimension. Secondly, as ω is closed it represents a (de Rham) cohomology class $[\omega] \in H^2(M)$.

Thirdly, with M being $2n$-dimensional and ω_p being non-degenerate for each $m \in M$, we know from section 1 that each ω_m^n is non-zero and thus ω^n is a non-vanishing $2n$-form representing an orientation: Symplectic manifolds are orientable. As ω^n is (the multiple of) a volume form, this implies that symplectomorphisms are volume preserving. Furthermore, for compact M, $\int_M \omega^n \neq 0$, implying $\int_M \omega \neq 0$ and by Stokes' theorem it follows that ω cannot be exact, i.e. $[\omega]$ is non-zero in cohomology.

EXAMPLES The first example we can come up with is the space $\mathbf{R}^{2n}$ with linear coordinates $\{q_1, p_1, \ldots, q_n, p_n\}$ and the standard symplectic structure $\omega = \sum_{j=1}^n dq_j \wedge dp_j = dq \wedge dp$. Linear coordinates on $\mathbf{R}^{2n}$ give a basis

$$\left\{ \left(\frac{\partial}{\partial q_1}\right)_m, \left(\frac{\partial}{\partial p_1}\right)_m, \ldots, \left(\frac{\partial}{\partial q_n}\right)_m, \left(\frac{\partial}{\partial p_n}\right)_m \right\}$$

for each $T_m M$ that is a symplectic basis with respect to ω_m. Charts with this property are called *symplectic charts*.

From the facts established in the previous paragraph, we quickly discover two manifolds that cannot carry a symplectic structure: Firstly the Möbius strip, which is non-orientable. Secondly, spheres S^n with $n \neq 2$ are compact and have $H^2(S^n) = 0$ – hence do not allow for a symplectic structure with $[\omega] \neq 0$.

For S^2, however, we can find a symplectic structure as every representative for non-zero elements of $H^2(S^2)$ is closed and non-degenerate. As this is quite a nice example, we will also take a look at the concrete symplectic structure: Think of S^2 as the set of unit vectors in $\mathbf{R}^3$. Then, for a $m \in S^2$, the tangent space at that point is the orthogonal complement (with respect to the standard inner product g) of the subspace spanned by $m \in S^2 \subset \mathbf{R}^3$ and we can define $\omega_m(v, w) = g(m, v \times w)$. This form is closed since S^2 is two-dimensional and it is also non-degenerate as for $v \neq 0$ we always have $\omega_m(v, m \times v) = g(m, v \times (m \times v)) = \det(m, v, m \times v) \neq 0$.

Apart from $S^2 = \mathbf{CP}^1$, all $\mathbf{CP}^n$ can be given a symplectic structure. Rather than showing this directly, the treatment of $\mathbf{CP}^n$ will be postponed until section 5, where we will be able to show that it is a symplectic manifold using symplectic reduction. Another very important class of examples is the following:

COTANGENT BUNDLES Let Q be a (smooth) n-dimensional manifold, then its cotangent bundle $T^*Q \xrightarrow{\pi} Q$ is a $2n$-dimensional manifold which locally has coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots p_n)$ where $q \in Q$ and $p \in (T_qQ)^*$. Now consider the bundle projection map π and its derivative:

$$\pi : T^*Q \longrightarrow Q \quad (q, p) \longmapsto q \qquad d\pi = \frac{\partial \pi}{\partial q} dq + \frac{\partial \pi}{\partial p} dp = dq.$$

Using this at a point (q, p), we can define a 1-form ϑ by $\vartheta_{(q,p)} = p \circ d_{(q,p)}\pi$:

$$\vartheta : T_{(q,p)}T^*Q \overset{d_{(q,p)}\pi}{\longrightarrow} T_qQ \overset{p}{\longrightarrow} \mathbf{R},$$

This form is defined independent of choice of coordinates and it is known as the *canonical 1-form.* We can, however work out ϑ in the coordinates given above. To do this we need to remember three facts: (i) $p \in T^*Q$ can be written as $p = \sum_{j=1}^n p_j dq_j$, (ii) $v \in TT^*Q$ can be written as $v = \sum_{k=1}^n (v_k \partial/\partial q_k + v_{n+k} \partial/\partial p_k)$ and (iii) $d_{(q,p)}\pi$ is zero on tangent vectors $\partial/\partial p_j$ that go along fibres and maps tangent vectors $\partial/\partial q_j$ to themselves. Putting these together gives

$$\begin{aligned} \vartheta_{(q,p)}(v) &= p \circ d_{(q,p)}\pi \left(\sum_{j=1}^n v_k \frac{\partial}{\partial q_k} + v_{n+k} \frac{\partial}{\partial p_k} \right) \\ &= \sum_{j,k=1}^n p_j dq_j \left(v_k \frac{\partial}{\partial q_k} \right) = \sum_{j=1}^n p_j v_j = pv. \end{aligned}$$

Thus $\vartheta = \sum_{j=1}^{n} p_j dq_j = pdq$ in local coordinates. We note that this way of writing ϑ is a fancy way of doing hardly anything. By differentiating $-\vartheta$, we get the *canonical* 2*-form*

$$\omega = -d\vartheta = -\sum_{j,k=1}^{n} \frac{\partial p_j}{\partial p_k} dp_k \wedge dq_j = \sum_{j=1}^{n} dq_j \wedge dp_j = dq \wedge dp.$$

It is immediate that ω is a symplectic structure for T^*Q as we have already managed to exhibit symplectic charts for it.

Another theorem on cotangent bundles says that given a diffeomorphism $f : Q \to R$, its lift $T^*f : T^*R \to T^*Q$ will actually be a symplectomorphism [1, Theorem 3.2.12]. and it turns out that T^* is a functor from the category of smooth manifolds and diffeomorphisms to the category of symplectic manifolds and symplectomorphisms.

PHYSICS Cotangent bundles arise frequently in physics, particularly in classical mechanics. If we have a mechanical system, it has a *configuration space* Q associated to it, containing all the possible constellations the system can take in space. This is fine for describing the spatial state of the system at one moment in time. However, physicists are also interested in describing future states of the system. To do this they need to consider the velocity or the momentum of the system as well.

This gives rise to the *phase spaces* TQ and T^*Q respectively. TM is also called *velocity phase space* as an element $(q, p) \in TQ$ gives the position p and a tangent vector p at that position encoding the velocity. T^*M is also known as *momentum phase space* as $(q, p) \in T^*Q$ gives with p a map that answers to every direction with a number. In the following we shall be dealing with momentum phase spaces only.

We can give several examples for this: Consider the harmonic oscillator. Its possible positions are expressed by a real line. Thus its momentum phase space is $T^*\mathbf{R} = \mathbf{R}^2$. Similarly, if we have a particle that can move freely in 3-space, its configuration space is $\mathbf{R}^3$ and the momentum phase space is $T^*\mathbf{R}^3 = \mathbf{R}^6$. Another example is given by a pendulum – its position in space can be described by an angle $\alpha \in [0, 2\pi]$, thus its configuration space is S^1 and its phase space is $T^*S^1 = S^1 \times \mathbf{R}$. Similarly the state of a double pendulumcan be described by two angles $\alpha, \beta \in [0, 2\pi]$ and thus the configuration space is T^2 with phase space $T^*T^2 = T^2 \times \mathbf{R}^2$.

We shall see some of these examples along with more notes on the relation of physics with symplectic geometry again in section 5.

VECTOR FIELDS Let (M, ω) be a symplectic manifold and $V(M)$ the *space of vector fields on* M, i.e. smooth sections of TM. Given a vector field, we can now use ω to get a 1-form by the map

$$\omega^\flat : V(M) \longrightarrow \Omega^1(M) \qquad X \longmapsto \iota_X \omega.$$

This map is linear and as ω is non-degenerate it is an isomorphism with inverse $\omega^\sharp$. We shall write $X^\flat$ and $\eta^\sharp$ instead of $\omega^\flat(X)$ and $\omega^\sharp(\eta)$ where no ambiguities concerning the symplectic structure in question can arise.

Given symplectic charts with local coordinates (q,p) it is useful to take $\eta \in \Omega^(M)$ and compute $\eta^\sharp$ explicitly. Given $\eta = \sum_{j=1}^n (a_j dq_j + b_j dp_j)$, we want to compute the coefficients of $\eta^\sharp = \sum_{j=1}^n (a'_j \partial/\partial q_j + b'_j \partial/\partial p_j)$ as defined implicitly by $\iota_{\eta^\sharp}\omega = \eta$:

$$
\begin{aligned}
\iota_{\eta^\sharp}\omega &= \sum_{j,k=1}^n dq_j \wedge dp_j \left(a'_k \frac{\partial}{\partial q_k} + b'_k \frac{\partial}{\partial p_k} \right) \\
&= \sum_{j,k=1}^n dq_j \left(a'_k \frac{\partial}{\partial q_k} + b'_k \frac{\partial}{\partial p_k} \right) dp_j - dp_j \left(a_k \frac{\partial}{\partial q_k} + b_k \frac{\partial}{\partial p_k} \right) dq_j \\
&= \sum_{j=1}^n (a'_j dp_j - b'_j dq_j) = \sum_{j=1}^n (a_j dq_j + b_j dp_j) = \eta
\end{aligned}
$$

Thus we get

$$
\eta^\sharp = \sum_{j=1}^n (b_j dq_j - a_j dq_j) \tag{B}
$$

Given a vector field X, we locally have a *flow* F_t associated to it and we remind ourselves of the *Lie derivative* $L_X Y = d/dt(F_{t*}Y)_{|t=0}$ of a vector field Y along the vector field X. This allows us to define the *Lie bracket* $[X,Y] = L_X Y$ that makes $(V(M), [\cdot,\cdot])$ a *Lie algebra*.

We can define the Lie derivative on forms using the pullback instead of the pushforward of the flow F_t: $L_X\eta = d/dt(F_t^*\eta)_{|t=0}$ for $\eta \in \Omega^j(M)$. For this case the relation

$$
L_X\eta = \iota_X d\eta + d\iota_X\eta \tag{C}
$$

as proved in [11, Proposition I.3.10] will be useful later. On functions $F \in C^\infty(M)$ the Lie derivative along X is defined as $L_X f = df(X)$. The latter is a derivation thus justifying the word 'derivative' in its name. Finally note that a function $F \in C^\infty(M)$ gives rise to a vector field $X_F = (dF)^\sharp$.

DARBOUX'S THEOREM The classification of symplectic manifolds is not as simple as the classification of symplectic vector spaces. However it is still possible to show that locally all symplectic manifolds of the same dimension are diffeomorphic. This is known as *Darboux's theorem*. The proof given below follows that of Moser and Weinstein that is given in many places in the literature such as [1, Theorem 3.2.2] or [8, 22.1]. It is remarked in [4, 2.1.4] that other proofs are possible and one of them is given there.

We start with a symplectic manifold (M,ω) and a point $m \in M$ for a neighbourhood of which we want to prove the theorem. First we note that by means of a chart φ, M is locally diffeomorphic to a vector space V with linear coordinates and we can arrange for $\varphi(m) = 0$. Thus we will just consider V and forms on V for the rest of this proof. With $\omega'_p = \omega_0$ for all $p \in V$, we can define the interpolation

$$[0,1] \longrightarrow \Omega^2(V) \qquad t \longmapsto \omega^t = (1-t)\omega + t\omega'$$

We know that for all $t \in [0,1]$ $\omega^t_0 = \omega'$ and thus all ω^t_0 are non-degenerate. As non-degeneracy is an open property, we can find an open ball B around 0 on all of which all of the ω^t are non-degenerate. As ω' and ω are closed, so is $\omega' - \omega$ and the Poincaré-Lemma now guarantees the existence of an $\eta \in \Omega^1(B)$ such that $\omega' - \omega = d\eta$. Since η is only determined up to a closed 1-form, it can be chosen such that $\eta_0 = 0$.

Next we use the non-degeneracy of the ω^t to define a vector field $X_t = \omega^{t\sharp}(-\eta)$ which in turn, on a sufficiently small neighbourhood, gives rise to a flow F_t, with $F_0 = \text{id}$. These flows can be associated to the Lie derivative $L.\cdot$ giving

$$\begin{aligned}
\tfrac{d}{dt}(F_t^*\omega^t) &= \tfrac{d}{dt}(F_t^*)\omega^t + F_t^*\tfrac{d}{dt}\omega^t && \text{product rule}\\
&= F_t^*(L_{X_t}\omega^t) + F_t^*(\omega' - \omega) && \text{relation with } L.\cdot\text{, compute}\\
&= F_t^*(\iota_{X_t}d\omega^t + d\iota_{X_t}\omega^t + \omega' - \omega) && \text{by equation (C)}\\
&= F_t^*(-d\eta + \omega' - \omega) && \omega \text{ closed, definition of } X_t\\
&= F_t^*(-\omega' + \omega + \omega' - \omega) && \text{construction of } \eta\\
&= 0.
\end{aligned}$$

Thus $F_t^*\omega^t$ is constant and $F_0 = \text{id}$ gives $F_1^*\omega^1 = F_1^*\omega' = \omega$ showing that F_1 locally is a diffeomorphism that pulls back the constant form ω' to the symplectic structure ω of our manifold.

It is noted in [1, §3.2] that this technique can also be used to prove the Morse lemma, as is done there or in [15, 3.9]. Although we would not expect to meet the Morse lemma here, on second thoughts it is not as remote as it first seems as it deals with putting a non-degenerate bilinear form (the Hessian) into a standard form as well.

Darboux's theorem shows that a symplectic structure is more restrictive than a Riemannian structure. Symplectic manifolds have no local invariants apart from their dimension, whereas Riemannian manifolds have many, such as the different flavours of curvature. Thus the study of symplectic manifolds is mainly a study of global properties.

3 MORE ON SYMPLECTIC MANIFOLDS

COMPLEX STRUCTURES A *complex structure* on a manifold is understood to be an atlas of charts going into $\mathbf{C}^n$ with holomorphic transition functions. Unfortunately this is not equivalent to the notion we get when we try to generalise the definition of a complex structure on a vector space as given on page 5. That generalisation will give us what is known as an *almost complex* *structure*, namely a section of the automorphism bundle $J \in \Gamma\mathrm{Aut}(TM)$ such that $J_m^2 = -\mathrm{id}_{T_mM}$ for every $m \in M$. A manifold with a (almost) complex structure is called a *(almost) complex manifold*.

For our purpose it is enough to keep in mind that given a complex manifold, its complex structure gives rise to an almost complex structure as shown in [3, §3]. A complex manifold can also carry a *hermitian structure* h which is a smooth section of the bundle of unitary automorphisms of the tangent spaces: $h \in \Gamma \bigsqcup_{m\in M} \mathrm{U}(T_mM, J_m)$.

Later we will use standard notation for complex manifolds, including vector fields $\partial/\partial z = 1/2(\partial/\partial x - i\partial/\partial y)$, $\partial/\partial \bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$, their dual 1-forms $dz, d\bar{z}$ and the operators $\partial, \bar{\partial}$ as introduced, say, in [3, §3].

Let M be a complex manifold with induced almost complex structure J and hermitian structure h, then $\omega = \mathrm{Im}(h)$ defines (twice) the *Kähler form* which is non-degenerate. If the Kähler form is closed, i.e. if (M, ω) is also a symplectic manifold, (M, J, ω) is called a *Kähler manifold*.

HAMILTONIAN VECTOR FIELDS AND FLOWS Consider a smooth function H on a symplectic manifold (M, ω) and the vector field X_H it implicitly defines via the equation $\iota_{X_H}\omega = dH$. In this case H is called a *Hamiltonian function* or *energy function* and X_H its *associated Hamiltonian vector field*. Different Hamiltonian functions can give rise to the same Hamiltonian vector field. However any two of these functions will only differ by a constant on each connected component of M. In a symplectic chart with coordinates (q, p) X_H can be given explicitly by equation (B):

$$X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} \frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial}{\partial p_j} \right).$$

The triple (M, ω, X_H) is called a *Hamiltonian system*. Hamiltonian systems are a huge topic of their own.

Given a vector field X on (M, ω), it is said to be *locally Hamiltonian* if for every point in M there is an open neighbourhood on which X is Hamiltonian. This is equivalent to locally $\iota_X\omega = dH$ being closed which is in turn equivalent to $L_X\omega = \iota_X d\omega + d\iota_X\omega = 0$. Thus it is also equivalent to the local flow F_t of X_H, also called the *Hamiltonian flow of* H, consisting of symplectomorphisms. The proof for this runs along the same lines as the argument and computation for the proof of Darboux's theorem.

POISSON BRACKETS Given smooth functions F, G on a symplectic manifold (M, ω) we define the *Poisson bracket on functions* using the Hamiltonian vector fields X_F and X_G induced by F and G:

$$\{F, G\} = \omega(X_F, X_G) = X_F^\flat(X_G) = (dF)^{\sharp\flat}(X_G) = dF(X_G) \qquad \text{(D)}$$

The Poisson bracket is bilinear, skew-symmetric and satisfies the Jacobi identity – thus it makes $(C^\infty(M), \{\cdot,\cdot\})$ a Lie algebra. By the last part of the equation above, we can relate the Poisson bracket on functions to the Lie derivative on functions: $\{F,G\} = dF(X_G) = L_{X_G}F$. Differentiating this expression gives $X_{\{F,G\}} = -[X_F, X_G]$ as shown with different sign conventions in [14, Proposition 3.6]. We can also define the Poisson bracket on 1-forms, to satisfy a similar relation:

$$\{\alpha,\beta\}^\sharp = -[\alpha^\sharp, \beta^\sharp] \qquad \alpha,\beta \in \Omega^1(M)$$

As shown in [2, §3.3, Theorems 1 and 3] the two definitions of the Poisson brackets are related by the exterior derivative: $d \circ \{\cdot,\cdot\} = \{\cdot,\cdot\} \circ (d \times d)$. Thus, the relations between the Poisson bracket on functions, the Poisson bracket on forms and the Lie bracket are summarised in the commutative diagram in figure 1.

$$\begin{array}{ccc}
C^\infty(M) \times C^\infty(M) & \xrightarrow{\{\cdot,\cdot\}} & C^\infty(M) \\
\Big\downarrow{\scriptstyle d\times d} & & \Big\downarrow{\scriptstyle d} \\
\Omega^1(M) \times \Omega^1(M) & \xrightarrow{\{\cdot,\cdot\}} & \Omega^1(M) \\
\Big\downarrow{\scriptstyle \sharp\times\sharp} & & \Big\downarrow{\scriptstyle \sharp} \\
V(M) \times V(M) & \xrightarrow{-[\cdot,\cdot]} & V(M)
\end{array}$$

FIGURE 1: Relating the the Poisson bracket on functions, the Poisson bracket on forms and the Lie bracket using d and $\sharp$.

Apart from establishing these abstract properties, we can also write the Poisson bracket in the coordinates (q,p) of a symplectic chart using equation (B):

$$\begin{aligned}
\{F,G\} &= dF((dG)^\sharp) \\
&= \sum_{j,k=1}^{n} \left(\frac{\partial F}{\partial q_j} dq_j + \frac{\partial F}{\partial p_j} dp_j\right)\left(\frac{\partial G}{\partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial G}{\partial q_k}\frac{\partial}{\partial p_k}\right) \\
&= \sum_{j=1}^{n} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}\right) \qquad \text{(E)}
\end{aligned}$$

Recall that for the Lie bracket $[X,Y] = 0$ means that the two vector fields X and Y 'commute', i.e. if running a bit along the flow of X and then along the flow of Y we will end up at the same point as when running along the flow Y first and that of X afterwards (see, for example [12, §V.1]). Similarly we can interpret to $\{F,G\} = dF(X_G) = 0$: it means that F is constant along the flow generated by G and vice versa. In particular, if we have a Hamiltonian function H, by antisymmetry of the Poisson bracket, we have $\{H, H\} = 0$. If H is a function giving the energy – as the alternate name given in the definition of Hamiltonian functions in the previous paragraph suggests – this equation encodes the conservation of energy.

GROUP ACTIONS We want to consider the smooth action of a Lie group G with Lie algebra $\mathbf{g} = T_eG$ on a symplectic manifold (M, ω). Here is some notation which we will use whenever talking about group actions: Let $\Phi : G \times M \to M$ be the group action and denote $\Phi(g, m)$ by $g.m$. Additionally we define the maps

$$\varphi_g : M \longrightarrow M \quad m \longmapsto g.m \qquad \text{and} \qquad \psi_m : G \longrightarrow M \quad g \longmapsto g.m$$

for the action of G on M as well as

$$\lambda_g : G \longrightarrow G \quad h \longmapsto gh \qquad \text{and} \qquad \rho_g : G \longrightarrow G \quad h \longmapsto hg$$

for the action of G on itself. The group action Φ is called *symplectic* if all of the φ_g are symplectomorphisms. A form $\alpha \in \Omega^j(M)$ is called *G-invariant* if $\varphi_g^*\alpha = \alpha$ for all $g \in G$. So in particular ω is G-invariant if and only if Φ is symplectic. Given a vector $\xi \in \mathbf{g}$, we can define a vector field $X_\xi^M \in V(M)$ associated to it using the exponential map*:

$$X_\xi^M : m \longmapsto (\psi_m)_{*e}\xi = \frac{d}{dt}\varphi_{\exp t\xi}(m)_{|t=o}$$

X_ξ^M is called the *infinitesimal generator of* ξ *on* M. If there is no ambiguity with respect to the manifold, we shall write simply X_ξ instead of X_ξ^M. For a fixed $\xi \in \mathbf{g}$, $X_\xi^\cdot$ is a natural transformation from the identity functor on the category of manifolds to T. If on the other hand we fix the manifold M, the map $\xi \mapsto X_\xi$ is a Lie algebra homomorphism $(\mathbf{g}, [\cdot, \cdot]) \to (V(M), [\cdot, \cdot])$. Φ is called *(weakly) Hamiltonian* if every X_ξ is a Hamiltonian vector field, i.e. if there is a function H_ξ such that $X_\xi^\flat = dH$.

Now assume Φ to be a Hamiltonian group action of an abelian group G on M. This implies that $\mathbf{g}$ is commutative with respect to the bracket. Now we can use the relation between the Lie and Poisson brackets and compute for $\xi, \zeta \in \mathbf{g}$

$$(d\{H_\xi, H_\zeta\})^\sharp = X_{\{H_\xi, H_\zeta\}} = -[X_\xi, X_\zeta] = -X_{[\xi,\zeta]} = -(dH_{[\xi,\zeta]})^\sharp.$$

Taking the first and the last expression and removing the $\sharp$ gives

$$c = \{H_\xi, H_\zeta\} - (-H_0)$$

*See [16, §2.4] for the construction of and facts on the exponential map.

where c is constant. As H_0 is constant as well, this implies that $\{H_\xi, H_\zeta\}$ is constant. If we assume G to be compact, then so are the closures of the integral curves generated by X_ξ as they are closed and contained in the compact image of some $\psi_{m'}$. Thus H_ζ takes a minimum at some point m on the closure of the integral curve generated by X_ξ giving $\{H_\xi, H_\zeta\}(m) = d_m H_\xi(X_{\zeta_m}) = 0$ implying

$$\omega(X_\xi, X_\zeta) = \{H_\xi, H_\zeta\} = 0 \tag{F}$$

everywhere in case all the objects in question exist. This result will be needed later.

Finally we note that, given any smooth manifold Q and an action Φ by a Lie group G on it, this action can be extended to a symplectic action on T^*Q as T^* lifts diffeomorphisms to symplectomorphisms. The extended action is given by

$$G \times T^*Q \longrightarrow T^*Q \qquad (g, (q, p_q)) \longmapsto (g.q, \varphi^*_{g^{-1}} p_q).$$

4 MOMENT MAPS

Using many of the definitions we have made so far, we are now in a position to define the notion of a moment map. This is a fairly technical and abstract notion but perseverance will be rewarded in the next section with nice results. The moment map can be thought of as an attempt to generalise the idea of a Hamiltonian function.

DEFINITION Let (M, ω) be a symplectic manifold and G a Lie group with Lie algebra $\mathbf{g}$ acting on M symplectically. Then a map

$$\mu : M \longrightarrow \mathbf{g}^*$$

with $\hat{\mu}$ denoting the induced map

$$\hat{\mu} : \mathbf{g} \longrightarrow C^\infty(M) \qquad \xi \longmapsto [m \mapsto \mu(m)(\xi)]$$

is called a *moment map* if for every $\xi \in \mathbf{g}$ the function $\hat{\mu}(\xi)$ is Hamiltonian, that is, if

$$d\hat{\mu}(\xi) = \iota_\xi \omega \qquad \text{or equivalently} \qquad X_{\hat{\mu}(\xi)} = X_\xi.$$

FACTS As the $\hat{\mu}(\xi)$ are Hamiltonian functions, those given by different μ can differ by at most a constant (on each connected component). Thus, any two moment maps differ only by a constant element of $\mathbf{g}^*$ on each connected component of M. Also, for moment maps to exist, the group action must be weakly Hamiltonian.

Another result that we get is the following: Let μ be a moment map along with all the setup required for it in the standard notation and a function

$H \in C^\infty(M)$ that is invariant under the group action, i.e. H is constant along orbits of the group action. Then we have

$$\mu(F_t(m)) = \mu(m) \qquad \text{for every } m \in M, t \in [0,1]$$

where F_t is the flow associated to X_H. A function H with this property is called an *integral* for the vector field X_H. To prove this, we pick the right formulas from our work on Poisson brackets on page 12, use the fact that H is constant along orbits of G as well as $X_\xi = (\iota_\xi \omega)^\sharp = (d\hat{\mu}(\xi))^\sharp = X_{\hat{\mu}(\xi)}$, to get, for any $\xi \in \mathbf{g}$,

$$0 = dH(X_\xi) = dH(X_{\hat{\mu}(\xi)}) = \{H, \hat{\mu}(\xi)\} = d\hat{\mu}(\xi)(X_H).$$

Ad AND Ad* Let Ad be the *adjoint representation* of G on its Lie algebra $\mathbf{g}$ coming from G acting on itself by conjugation

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathbf{g}) \qquad g \longmapsto \left[\xi \mapsto (\rho_g^{-1}\lambda_g)_{*_e}\xi\right];$$

and let Ad* be its dual, the *coadjoint representation* on $\mathbf{g}^*$

$$\mathrm{Ad}^* : G \longrightarrow \mathrm{Aut}(\mathbf{g}^*) \qquad g \longmapsto \left[\tau \mapsto (\mathrm{Ad}_{g^{-1}})^*\tau\right].$$

Note, that Ad and Ad* are constant $\mathrm{id}_\mathbf{g}$ and $\mathrm{id}_{\mathbf{g}^*}$ respectively if G is abelian as then, for every $g \in G$, $\mathrm{Ad}_g = (\rho_g^{-1}\lambda_g)_{*_e} = \mathrm{id}_\mathbf{g}$. We will later need the following result on how the vector field associated to a vector $\xi \in \mathbf{g}$ is related to that associated to $\mathrm{Ad}_g\xi$:

$$\begin{aligned}
(X_{\mathrm{Ad}_g\xi})_m &= \frac{d}{dt}\varphi_{\exp t\mathrm{Ad}_g\xi}(m)_{|t=0} \\
&= \frac{d}{dt}\varphi_{g(\exp t\xi)g^{-1}}(m)_{|t=0} = \frac{d}{dt}\varphi_g\varphi_{\exp t\xi}(g^{-1}.m)_{|t=0} \\
&= \varphi_{g_*}\left(X_{\xi_{g^{-1}.m}}\right)
\end{aligned}$$

Finally, let Φ be a symplectic action of G on M and μ be a moment map for that group action. Then μ is said to be Ad*-*equivariant* if the diagram in figure 2 commutes.

$$\begin{CD}
M @>{\varphi_g}>> M \\
@V{\mu}VV @VV{\mu}V \\
\mathbf{g}^* @>{(\mathrm{Ad}^*)_g}>> \mathbf{g}^*
\end{CD}$$

FIGURE 2: μ is Ad*-equivariant if this diagram commutes.

To see that Ad*-equivariance is an important property let μ be an Ad*-equivariant moment map, $\tau \in \mathbf{g}$ and $G_\tau = \{g \in G | (Ad^*)_g \tau = \tau\}$ the *isotropy subgroup of* τ *in* G. G_τ is closed, thus a Lie group. We see that with $g \in G_\tau$ and $m \in \mu^{-1}(\tau)$

$$\mu(g.m) = \mu(\varphi_g(m)) = (Ad^*)_g \mu(m) = (Ad^*)_g \tau = \tau. \tag{G}$$

Thus Ad*-equivariance ensures that G_τ acts on $\mu^{-1}(\tau)$. In the next paragraph we will see an example for an Ad*-equivariant moment map.

COTANGENT BUNDLE Given a manifold Q with a group action Φ by G we have seen that this extends to a symplectic group action Φ' on T^*Q. We can now define

$$\mu : T^*Q \longrightarrow \mathbf{g}^* \qquad (q,p) \longmapsto \left[\xi \mapsto \left(\vartheta(X_\xi^{T^*Q}\right)_{(q,p)}\right]$$

where ϑ is the canonical 1-form as defined on page 7. Since G acts by symplectomorphisms on T^*Q, ω and thus ϑ are constant along orbits, using equation (C), we get

$$0 = L_{X_\xi}\vartheta = d(\iota_{X_\xi}\vartheta) + \iota_{X_\xi} d\vartheta = d(\hat{\mu}(\xi)) - \iota_{X_\xi}\omega$$

so μ is a moment map. A computation using G-invariance of ϑ and a fair amount of rewriting shows that μ is Ad*-equivariant: for any $m = (q,p) \in T^*Q, g \in G$ and $\xi \in \mathbf{g}$ we have

$$\begin{aligned}
((\mathrm{Ad}^*)_g \mu(m))(\xi) &= \mu(m)(\mathrm{Ad}_{g^{-1}}\xi) \\
&= \left(\vartheta(X^{T^*Q}_{\mathrm{Ad}_{g^{-1}}\xi})\right)_m = \vartheta_m\left(({\varphi_{g^{-1}}}_*)_{g.m}(X_\xi^{T^*Q})_{g.m}\right) \\
&= \left(\vartheta(X_\xi^{T^*Q})\right)_{g.m} = \mu(\varphi_g(m))(\xi).
\end{aligned}$$

A further computation using the naturality of X_ξ on the bundle projection π and its pushforward reveals a less abstract expression for this map:

$$\begin{aligned}
\mu(q,p)(\xi) &= \vartheta(X_\xi^{T^*Q})_{(q,p)} = p\left(dp(X_\xi^{T^*Q})\right)_{(q,p)} \\
&= p\left(d\pi(X_\xi^{T^*Q})\right)_{(q,p)} = p\left(\pi_* X_\xi^{T^*Q}\right)_{(q,p)} \\
&= p(X^Q_{\xi\, q}).
\end{aligned}$$

EXAMPLES AND PHYSICS As the name 'moment map' suggests, the main examples for moment maps come from physics – although of course physicists have a more 'hands-on' approach to this and tend to use neither the abstract definitions we have seen nor the name 'moment map' but more concrete descriptions as 'energy' or 'momentum'. We will recruit our examples from those given in the paragraph on cotangent bundles in section refmanifolds.

HARMONIC OSCILLATOR We start with the example of the *harmonic oscillator*. Its phase space was seen to be $\mathbf{R}^2 = T^*\mathbf{R}$ with coordinates q for the 'location' and p for the 'momentum' and symplectic form $\omega = dq \wedge dp$. By the isomorphism defined by $q \mapsto 1$ and $p \mapsto i$, we can identify the phase space and similarly its tangent spaces with $\mathbf{C}$ and define a S^1-action on it by multiplication with $e^{i\alpha}$ with $\alpha \in [0, 2\pi]$.

This is in fact a symplectic group action as rotations of the plane are volume preserving and ω is a volume form. We now define a map

$$\mu : M \to \mathbf{R} \qquad (q,p) \mapsto 1/2(q^2 + p^2)$$

and identify $\mathbf{s^1}^* = (i\mathbf{R})^* \simeq \mathbf{R}$. This map is the energy of the harmonic oscillator and the group action is a change of coordinates from straightforward position-and-momentum coordinates to mixed ones where each coordinate carries partial information on both the position and the momentum.

To see that our map μ is a moment map first observe that the vector fields associated to vectors $iy \in \mathbf{s^1}$ are

$$(X_{iy})_{re^{i\alpha}} = \frac{d}{dt}\varphi_{\exp tiy}(re^{i\alpha})_{|t=0} = \frac{d}{dt}re^{i(ty+\alpha)}{}_{|t=0} = irye^{i(ty+\alpha)}{}_{|t=0} = irye^{i\alpha}$$

where $(q,p) = re^{i\alpha} \in T^*Q$. With the proper identifications of the different bases this gives

$$(X_{iy})_{re^{i\alpha}} = irye^{i\alpha} = ry(i\cos\alpha - \sin\alpha) = y(-p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p}).$$

Eventually we see that

$$\begin{aligned} \iota_{X_{iy}}\omega &= ydq \wedge dp\left(-p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p}\right) \\ &= y(qdq - pdp) = \frac{y}{2}d(q^2 + p^2) = d(\hat{\mu}(iy))(q,p), \end{aligned}$$

proving that μ is a moment map. Since S^1 is abelian, $\mathrm{Ad}_g = \mathrm{id}_{\mathbf{g}}$ for every $g \in G$ and we see that μ is Ad^*-equivariant:

$$\begin{aligned} ((\mathrm{Ad}^*)_{e^{i\alpha}}\mu(re^{i\beta}))(iy) &= \mu(re^{i\beta})(\mathrm{id}_{\mathbf{g}}iy) \\ &= yr^2 = \mu(re^{i(\alpha+\beta)})(iy) = \mu(\varphi_{e^{i\alpha}}(re^{i\beta}))(iy) \end{aligned}$$

MOMENTUM Next, we proceed to higher dimensions and consider $\mathbf{R}^3$ with linear coordinates q and $\mathbf{R}^3$ acting on it by translations, i.e. for $g \in \mathbf{R}^3$ we have $\varphi_g(q) = q + g$. As we know this action extends to a symplectic action on $T^*\mathbf{R}^3$ given by $\varphi'_g(q,p) = (q+g, \varphi^*_{-g}p)$. By our general notes on the cotangent bundle we can now compute an Ad^*-equivariant moment map

$$\mu(q,p)(\xi) = p(X_\xi^{\mathbf{R}^3})_q = p\left(\frac{d}{dt}\varphi_{\exp t\xi}(q)_{|t=0}\right) = p\left(\frac{d}{dt}(q+t\xi)_{|t=0}\right) = p(\xi).$$

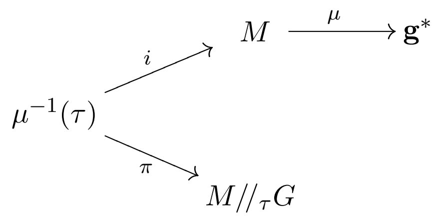

FIGURE 3: Maps involved in getting the symplectic quotient.

This example probably illustrates best how the moment map got its name. Given the group action by translation which represents the idea that at every point in space physics are the same we get a moment map that gives us – the momentum.

The angular momentum arises in a similar fashion by the usual action of SO(3) on $\mathbf{R}^3$. It is given by the moment map $\mu(q,p)(\xi) = q \times p$ where we think of $q \times p$ as a vector of $\mathbf{so}(3)^*$ via the identification of $\mathbf{so}(3)$ with $\mathbf{R}^3$ and duality induced by the standard inner product on $\mathbf{R}^3$. Actually computing this is rather lengthy as it requires a good deal of care when working out the vector fields X_ξ for $\xi \in \mathbf{so}(3)$.

5 SYMPLECTIC REDUCTION

So far we have seen quite a few definitions and facts with a hint of physics in them. These preparations are about to bear fruit as they are the correct setting for the process of *symplectic reduction* that enables us to go from one symplectic manifold to another, lower-dimensional symplectic manifold. This process has first been described by Marsden and Weinstein in [13], hence the alternative name *Marsden-Weinstein reduction.* Their text already contains most of the points that are covered in newer textbooks treating this topic, including the examples given here.

DEFINITION / THEOREM Our setting is the following: Let (M, ω) be a symplectic manifold, G a Lie group acting on it, μ an Ad*-equivariant moment map for this group action and τ a regular value of μ. Then the preimage $\mu^{-1}(\tau)$ is a submanifold with the isotropy subgroup G_τ of G acting on it as shown in equation (G). Assuming that this action is free and properly discontinuous, the quotient $\mu^{-1}(\tau)/G_\tau$ is also a manifold. This is often denoted $M//_\tau G$ and called the *Marsden-Weinstein quotient* or *symplectic quotient.*

We will now prove the main fact about this manifold, namely that it is a symplectic manifold with a symplectic structure ω' given by $\pi^*\omega' = i^*\omega$ with the maps as illustrated in figure 3, i.e.

$$\omega'_{[m]}([v],[w]) = \omega_m(v,w) \qquad \text{for } m \in \mu^{-1}(\tau),\ v,w \in T_m\mu^{-1}(\tau).$$

where we note that, using the definition of a moment map, for all $\xi \in \mathbf{g}$

$$T_m\mu^{-1}(\tau) = \ker d_m\mu = \ker d_m\hat{\mu}(\xi) = \ker \omega_m(X_\xi, \cdot) = T_m(G.m)^\omega,$$

giving that the tangent space at any $[m] \in M//_\tau G$ is

$$T_{[m]}M//_\tau G = T_m\mu^{-1}(\tau)/T_m(G.m) = T_m\mu^{-1}(\tau)/(T_m\mu^{-1}(\tau))^\omega. \tag{H}$$

With two different kinds of equivalence classes involved in the definition of ω', we have to check for well-definedness. With the previous result it is immediate that for $v, w \in T_m\mu^{-1}(\tau)$ and $u \in T_m(G_\tau.m) \subset T_m(G.m)$ we have $\omega_m(v, u) = 0$ and thus $\omega_m(v, w + u) = \omega_m(v, w)$. A similar computation to that in equation (G) shows that $\omega_{g.m}(v, w) = \omega_m(v, w)$ for $g \in G_\tau$.

Next, suppose that $\omega'_{[m]}([v], [w]) = \omega_m(v, w) = 0$ for all $v \in T_m\mu^{-1}(\tau)$. Then $w \in (T_m\mu^{-1}(\tau))^\omega = T_m(G.m)$ and thus $[w] = 0$, showing that ω' is non-degenerate. Finally, we can see by the properties of the quotient projection and inclusion maps that $d\pi^*\omega' = di^*\omega = i^*d\omega = 0$ implying $\pi^*d\omega' = 0$ and, as π is a submersion, $d\omega' = 0$, completing the proof that ω' is a symplectic structure on $M//_\tau G$.

Now that we have seen that everything is well-defined, the last part of equation (H) implies that $\mu^{-1}(\tau)$ is a coisotropic submanifold, i.e. a submanifold all tangent spaces of which are coisotropic in the tangent spaces of the surrounding manifold. In [14, §5.3] the symplectic quotient is introduced using the setting of coisotropic submanifolds.

HARMONIC OSCILLATOR AND COMPLEX PROJECTIVE SPACE As promised before we will use the technique of symplectic reduction to show that $\mathbf{CP}^n$ is a symplectic manifold in an unusual way. To do this, consider $\mathbf{C}^{n+1}\setminus\{0\}$ as a symplectic manifold with the standard symplectic structure for $\mathbf{C}^{n+1}$. Let S^1 act on it by scalar multiplication with $e^{i\alpha}$. This setting can be thought of as a system of $(n+1)$ harmonic oscillators. Thus their total energy function $\mu : (z_0, \dots, z_n) \mapsto 1/2 \sum_{j=0}^n |z_j|^2$ is an Ad^*-equivariant moment map.

As 0 is not in our original manifold, every $\tau \in \mathbf{R} \simeq \mathbf{s^1}^*$ is a regular value and its preimage is a $(2n+1)$-sphere of radius $\sqrt{2\tau}$ in $\mathbf{C}^{n+1}$. Since S^1 is abelian, the isotropy subgroup S^1_τ is all of S^1. Going the last step of the reduction process, we arrive at $\mathbf{C}^{n+1}//_\tau S^1 = \sqrt{\tau}S^{2n+1}/S^1$. Thus $\mathbf{CP}^n$ is a symplectic manifold.

It turns out that for $\tau = 1/2$ the symplectic structure we get in this way is the same as the symplectic structure induced by the *Fubini-Study metric* which is derived, for example, in [11, Example IX.6.2]. Note, that the symplectic structure we get from the reduction process depends on the choice of τ.

For $n = 0$, this is just the example of the single harmonic oscillator we have seen before. Exploiting the moment map and the S^1-symmetry, we are able to reduce the complex plane to a single point. This tells us that once we know the energy of a harmonic oscillator, its future behaviour can be determined from an initial 'position'.

MORE EXAMPLES Similarly to the harmonic oscillator we can treat the momentum: Every value of the moment map $\mu(q,p) = p$ is regular and $\mu^{-1}(\tau) = \mathbf{R}^3 \times \{\tau\}$. As the isotropy subgroup is all of $\mathbf{R}^3$ and the action by translation on the configuration space is free and properly transitive, we can proceed and this space is reduced to a point as well – telling us that once we know the position and momentum of a moving object in space, we can determine its future behaviour.

We shall not work out the reduction induced by the angular momentum here as it is by far more complicated than the previous examples: finding preimages for $\mu(q,p) = q \times p$ and describing them precisely is far from being obvious and then finding the isotropy subgroup requires extra attention as well since SO(3) is not abelian. The preimages for regular values $\tau \neq 0$ will be 3-dimensional as we can choose q out of a plane and then have 'half a circle"s worth of choices for p. The isotropy subgroups are copies of S^1 in SO(3). Unlike the previous examples we do not reduce by once the dimension of the group when taking the preimage and once again the dimension of the group when dividing by all of it, but the dimensions are split differently in the process, still adding up to twice the dimension of the group.

In [13, §6] there are more examples for symplectic reduction, the last of which outlines how it can also be used in general relativity.

PHYSICS After having seen these physical examples it is a good point to remark that the process of symplectic reduction is a geometric way to think of the finding of 'symmetries' and 'invariants' in physics and the elimination of variables based on those symmetries. Furthermore the process of symplectic reduction gives a formal geometric backing for what the physicists do.

More precisely, physicists tend to consider systems described by a *Lagrangian function* $L(q, \dot{q}, t)$ where $\dot{q}$ denotes the derivative of q with respect to t and are interested in those paths q between two given points $q(0)$ and $q(1)$ in space that minimise the integral $\int_0^1 L(q(t), \dot{q}(t), t)dt$. A computation as done in [14, Lemma 1.1] shows that such paths satisfy the *Euler-Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} \qquad \text{for } 1 \leq j \leq n.$$

Instead of using this system of n second order differential equations, we introduce new variables

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \qquad \text{for } 1 \leq j \leq n.$$

With these we define the *Hamiltonian function* $H(q,p,t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q, \dot{q}, t)$. This kind of Hamiltonian function is the origin for the name of Hamiltonian functions in section 3. Differentiating H and using the Euler-Lagrange equations as well as the definition of the p_j, we get

$$\frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\dot{p}_j \qquad \text{and} \qquad \frac{\partial H}{\partial p_j} = \dot{q}_j \qquad \text{for } 1 \leq j \leq n.$$

These are known as the *Hamilton equations* and they constitute a system of $2n$ ordinary differential equations.

The process of going from the $\dot{q}_j$ to the p_j is called *Legendre transformation* and from a formal point of view it takes us from the velocity phase space TQ to the momentum phase space T^*Q. This is where the symplectic structure is hiding. We also note that the fact that symplectic reduction reduces by an even number of dimensions fits in well as we cannot simply remove a dimension in some q_j without removing the corresponding p_j.

OUTLOOK The process of symplectic reduction as it is described here has been generalised for similar constructions on Kähler or hyperkähler (three compatible complex and symplectic structures) manifolds. In each case there are stronger restrictions on the group action which has to preserve all of the structures. In the hyperkähler case, moment maps for all three symplectic structures are used and the reduction reduces by more (twice as many) dimensions. A brief account of these types of reduction along with further references is given in [17].

6 SPECIAL LAGRANGIAN SUBMANIFOLDS

LAGRANGIAN SUBSPACES Recall that in section 1 we defined the notion of a *Lagrangian subspace* W of a $2n$-dimensional symplectic vector space (V, ω) by the condition $W^\omega = W$. This is equivalent to saying that W is n-dimensional and $\omega_{|W} = 0$.

A more intuitive way to think about this is given by Harvey and Lawson in [9, §III.1] – but it requires a slightly less general setting: Consider the space $\mathbf{C}^n$ with its normal complex structure J, real inner product g, hermitian inner product H as well as the standard symplectic form ω. Then we consider subspaces W with $\dim_{\mathbf{R}} W = n$. These are called *totally real* if they do not contain any complex lines, i.e. if $v \in W \Rightarrow Jv \notin W$ or if $\dim_{\mathbf{C}}(\operatorname{span}_{\mathbf{C}} W) = n$.

We may want W to be even 'less complex' by imposing the condition that for $v \in W$ we have $g(Jv, w) = 0$ for all $w \in W$, thus requiring W to be 'orthogonal' to the complex structure. A short computation involving equation (A),

$$0 = g(Jv, \cdot)_{|W} = \omega(Jv, J\cdot)_{|W} = \omega(v, \cdot)_{|W},$$

reveals that this condition is actually the condition for a Lagrangian subspace and hence allows us to think of Lagrangian subspaces as very 'non-complex' subspaces. We will denote the set of oriented Lagrangian subspaces by $\mathrm{Lag}(n)$. Given $W \in \mathrm{Lag}(n)$ and $U \in \mathrm{U}(n)$, we have

$$\omega(U(W), U(W)) = \mathrm{Im}\ H(U(W), U(W)) = \mathrm{Im}\ H(W, W) = \omega(W, W) = 0$$

and thus $\mathrm{U}(n)$ acts on $\mathrm{Lag}(n)$. Let $\mathbf{b}$ be an orthonormal basis for W. Then, W being Lagrangian implies that $\mathbf{b} \cup J\mathbf{b}$ is an orthonormal basis for $\mathbf{C}^n$ and equally that $\mathbf{b}$ is in fact a unitary basis for $\mathbf{C}^n$. As $\mathrm{U}(n)$ acts transitively on the set of unitary bases for $\mathbf{C}^n$, it also acts transitively on $\mathrm{Lag}(n)$. Since we required the elements of $\mathrm{Lag}(n)$ to be oriented, the isotropy subgroup of this action at $\mathbf{R}^n \times 0$ is

$$\mathrm{U}(n)_{\mathbf{R}^n \times 0} = \{U \in \mathrm{U}(n) | U(\mathbf{R}^n \times 0) = \mathbf{R}^n \times 0\} = \mathrm{SO}(n)$$

where we think of $\mathrm{SO}(n)$ as a subgroup of $\mathrm{U}(n)$ by letting it act diagonally on $\mathbf{R}^n \times \mathbf{R}^n$. Hence

$$\mathrm{Lag}(n) \simeq \mathrm{U}(n)/\mathrm{SO}(n) \tag{I}$$

In this setting, given an element $U \in \mathrm{U}(n)$, it represents the Lagrangian subspace $W = U(\mathbf{R}^n \times 0)$ with the orientation induced by the standard orientation on $\mathbf{R}^n$.

SPECIAL LAGRANGIAN SUBSPACES Now we can define the set of *special Lagrangian subspaces* to be

$$\mathrm{SLag}(n) = \det_{\mathbf{C}}^{-1}(1) \cap \mathrm{Lag}(n) \simeq \mathrm{SU}(n)/\mathrm{SO}(n),$$

where the first part is well-defined since $\det_{\mathbf{C}}$ is invariant under the $\mathrm{SO}(n)$-action described in the previous paragraph and the latter part arises by a similar argument to that leading to equation (I).

As $\det_{\mathbf{C}}(\mathrm{Lag}(n)) = S^1 \subset \mathbf{C}$, $\mathrm{Im}(\det_{\mathbf{C}} W)$ is zero if and only if $\det_{\mathbf{C}} W = \pm 1$. Identifying both copies of each subspace W with opposite orientations, we get $W \in \mathrm{SLag}(n)$ if and only if $\mathrm{Im}(\det_{\mathbf{C}} W) = 0$ for some orientation of W.

We can slightly generalise the notion of a special Lagrangian subspace to a special Lagrangian subspace *with phase* $e^{i\alpha}$ which has determinant $\pm e^{i\alpha}$ instead of ± 1. It is noted in [9, §III.1] that the geometries arising from phased subspaces are equivalent to those with phase 0 under $\mathrm{U}(n)$ – thus this generalisation does not do any harm and we shall omit the $e^{i\alpha}$ in the notation. Using this generalisation we have that any given complex volume form is of the form $\Omega = e^{i\alpha} \det_{\mathbf{C}}$ and will thus use $\mathrm{Im}\ \Omega = 0$ as the condition for a Lagrangian subspace to be special Lagrangian with respect to (the phase of) Ω. This terminology is in line with that used in [7] and [5].

MANIFOLDS We now transfer the notion of Lagrangian and special Lagrangian subspaces to submanifolds by saying that a submanifold is *(special) Lagrangian* if all of its tangent spaces are (special) Lagrangian subspaces of the tangent spaces of the ambient manifold. More concretely – a submanifold N of a symplectic manifold (M, ω) is Lagrangian if

$$\omega_{|N} = 0$$

and it is special Lagrangian if, with a holomorphic volume form Ω, we have

$$\omega_{|N} = 0 \qquad \text{and} \qquad \text{Im } \Omega_{|N} = 0. \tag{J}$$

CALABI-YAU MANIFOLDS Before proceeding, we consolidate all the different structures we use into a single notion. Let (M, ω) be Kähler manifold and Ω a holomorphic volume form. Then (M, ω, Ω) is called an *almost Calabi-Yau manifold*. In this setting the notion of a special Lagrangian submanifold in the sense of equation (J) is still meaningful. If $\omega^n/n! = c\,\Omega \wedge \bar{\Omega}$ with a constant c, we call (M, ω, Ω) a *Calabi-Yau manifold*.

CALIBRATIONS Harvey and Lawson's text [9] is concerned with calibrated geometries. A *calibration* is a closed k-form η such that $\eta_{|W} \leq \text{vol}_M$ for any k-dimensional subspace W of one of the tangent spaces. We will be dealing with , Riemannian manifolds M with a calibration η. A k-dimensional submanifold $N \subset M$ with $\eta_{|N} = \text{vol}_N$ is called a *calibrated submanifold* and it is of minimal volume in its homology class. For compact calibrated N and another N' from the same homology class this is proved by

$$\text{Vol}(N) = \int_N \text{vol}_N = \int_N \eta = \int_{N'} \eta \leq \int_{N'} \text{vol}_{N'} = \text{Vol}(N').$$

Given a Calabi-Yau manifold (M, ω, Ω), Re Ω is a calibration and thus special Lagrangian submanifolds $N \subset M$ are volume minimising in their homology class. It is noted in [9, §III.2.D] that to a certain the converse is true as well.

EXAMPLES Given a closed 1-form η on $\mathbf{R}^n$, its graph $\Gamma_\eta = \{(q, \eta_q) | q \in \mathbf{R}^n\}$ in the cotangent bundle $T^*\mathbf{R}^n \simeq \mathbf{C}^n$ is a Lagrangian submanifold as $\omega_{|\Gamma_\eta} = dq \wedge dp_{|\Gamma_\eta} = dq \wedge d\eta_{|\Gamma_\eta} = 0$. By [9, Theorem 2.3] Γ_η is special Lagrangian if locally at Γ_η there exists a function F such that $\eta = dF$ and $\text{Im}[\det_{\mathbf{C}}(\text{id} + i\text{Hess } F)] = 0$.

Before proceeding to the next example recall the definition of the Poisson bracket in equation (D) as $\{F, G\} = \omega(X_F, X_G)$: a regular submanifold $N = \{m \in \mathbf{C}^n | F_1(m) = \ldots F_l(m) = 0\} \subset \mathbf{C}^n$ of codimension l is Lagrangian if $\{F_j, F_k\}_{|N} = 0$ for $1 \leq j, k \leq l^\dagger$.

† This is the same condition as that for the Hamiltonian system associated to the F_j to be integrable.

Equation (E) gives an expression for the Poisson bracket with respect to $\partial/\partial x_j$ and $\partial/\partial x_j$ coming from the standard symplectic charts which we can transfer into an expression using $\partial/\partial z$ and $\partial/\partial \bar{z}$:

$$\begin{aligned}
\{F,G\} &= \sum_{j=1}^{n}\left(\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y_j}-\frac{\partial F}{\partial y_j}\frac{\partial G}{\partial x_j}\right)\\
&= \sum_{j=1}^{n}\left(\frac{\partial F}{\partial \bar{z}_j}\frac{\partial G}{\partial y_j}+\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial y_j}+i\frac{\partial F}{\partial \bar{z}_j}\frac{\partial G}{\partial x_j}-i\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial x_j}\right)\\
&= i\sum_{j=1}^{n}\left(-\frac{\partial F}{\partial \bar{z}_j}\frac{\partial G}{\partial \bar{z}_j}+\frac{\partial F}{\partial \bar{z}_j}\frac{\partial G}{\partial z_j}-\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial \bar{z}_j}+\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial z_j}\right.\\
&\qquad\left.+\frac{\partial F}{\partial \bar{z}_j}\frac{\partial G}{\partial \bar{z}_j}+\frac{\partial F}{\partial \bar{z}_j}\frac{\partial G}{\partial z_j}-\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial \bar{z}_j}-\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial z_j}\right)\\
&= 2i\sum_{j=1}^{n}\left(\frac{\partial F}{\partial \bar{z}_j}\frac{\partial G}{\partial z_j}-\frac{\partial F}{\partial z_j}\frac{\partial G}{\partial \bar{z}_j}\right)
\end{aligned}$$

Using this computation, we can look into another example which is adapted from [9, §III.3.A]: consider the map

$$f:\mathbf{C}^n\to\mathbf{R}^n \quad (z_1,\dots z_n)\mapsto(|z_n|^2-|z_1|^2,\dots,|z_n|^2-|z_{n-1}|^2,\mathrm{Re}[i^n z_1\dots z_n])$$

which is regular everywhere away from $0\in\mathbf{C}^n$. Now consider preimages of non-zero vectors $v\in\mathbf{R}^n$. These $f^{-1}(v)$ are submanifolds which are defined by equations $f_1'=\ldots=f_n'=0$ where $f_j'=f_j-v_j$. Their partial derivatives are:

	$\partial f_j'/\partial z_k$	$\partial f_j'/\partial \bar{z}_k$
$j<n, k<n$	$-\delta_{jk}\bar{z}_j$	$-\delta_{jk}z_j$
$j<n, k=n$	$\bar{z}_n$	z_n
$j=n, k\le n$	$i^n z_1\dots\widehat{z_k}\dots z_n$	$\overline{i^n}\bar{z}_1\dots\widehat{\bar{z}_k}\dots\bar{z}_n$

Using all of these we get

$$\{f_j,f_k\}=\begin{cases}2i\sum_{l=1}^{n}\delta_{jl}\delta_{kl}(z_j\bar{z}_k-\bar{z}_jz_k)\\ \qquad=2i\delta_{jk}(z_j\bar{z}_k-\bar{z}_jz_k)=0 & j,k<n\\ 2i\sum_{l=1}^{n}\delta_{jl}(z_j i^n z_1\dots\widehat{z_j}\dots z_n\\ \qquad-\bar{z}_j i^n\bar{z}_1\dots\widehat{\bar{z}_j}\dots\bar{z}_n)=0 & j<n,k=n\\ 0 \text{ by antisymmetry} & j=n,\end{cases}$$

proving that N is a Lagrangian submanifold. It is remarked in [9, §III.2.C] that in the general setting of a submanifold implicitly defined by equations F_j we can consider the complex linear map L mapping the standard basis for $\mathbf{C}^n$ to the basis $\{i\partial F_1/\partial \bar{z}, \dots, i\partial F_n/\partial \bar{z}\}$ of the tangent space of N.It allows us to determine whether N is special Lagrangian or not by checking whether or not $\mathrm{Im}[\det_{\mathbf{C}} L] = 0$. Applying this to our example gives

$$\begin{aligned}
\det_{\mathbf{C}} \left(i\frac{\partial f_j}{\partial \bar{z}_k} \right)_{jk} &= i^n \det_{\mathbf{C}} \begin{bmatrix} -z_1 & & & z_n \\ & \ddots & & \vdots \\ & & -z_{n-1} & z_n \\ i^n \widehat{\bar{z}_1} \dots \bar{z}_n & & \cdots & i^n \bar{z}_1 \dots \widehat{\bar{z}_n} \end{bmatrix} \\
&= (-1)^n \sum_{j=1}^{n} \pm \bar{z}_1 \dots \widehat{\bar{z}_j} \dots \bar{z}_n \, z_1 \dots \widehat{z_j} \dots z_m \\
&= (-1)^n \sum_{j=1}^{n} \pm |z_1 \dots \widehat{z_j} \dots z_n|^2 \in \mathbf{R},
\end{aligned}$$

proving that the submanifold N is special Lagrangian. We will work a bit more with this example after the next paragraph. More examples are given in [9, §III.3] and in [10, §8].

FACTS The following constructions are from [5] and [7], going mainly along the more elegant presentation of the latter. They show that the notions of special Lagrangian submanifolds and symplectic reduction are as 'compatible' as we could hope for: Given the setup for symplectic reduction of an alomost Calabi-Yau manifold (M, ω, Ω), Ω will induce a form Ω' on the reduced space; special Lagrangian submanifolds of which with respect to Ω' can be lifted to special Lagrangian submanifolds in M and in fact, any T^l-invariant special Lagrangian submanifold of M does arise in this way.

Let (M, ω, Ω) be an almost Calabi-Yau manifold with a T^l-action preserving both forms, let μ be a moment map for that action, $\tau \in \mathbf{t^l}$ a regular value of μ and $\xi_1, \dots \xi_l$ be a basis for the Lie algebra $\mathbf{t^l}$.

The first thing to see is that Ω induces a nowhere-vanishing $n-l$-form on reduced spaces $M//_\tau G$ defined by $\pi^*\Omega' = \iota_{X_{\xi_1} \dots X_{\xi_l}} \Omega$. To prove it, recall from equation (H) that $T_{[m]}(M//_\tau T^l) = T_m \mu^{-1}(\tau)/T_m(T^l.m)$ which implies that

$$\iota_{X_{\xi_1} \dots X_{\xi_l}} \Omega|_{T_m(T^l.m)} = 0$$

for every $m \in \mu^{-1}(\tau)$. As Ω is T^l-invariant, so is Ω' and thus we have that Ω' is a well defined $(n-l)$-form on $M//_\tau G$. The non-degeneracy and is inherited from Ω by the construction.

Secondly, we can establish that special Lagrangian submanifolds N' of $(M//_\tau T^l, \omega', \Omega')$ lift to special Lagrangian submanifolds $N = i(\pi^{-1}(N')) \subset M$ with the maps i and π as in figure 3: Let $\mathbf{b}$ be a basis for $T_{[m]}M//_\tau T^l$, then

$$\pi^{-1}(\mathbf{b}) \cup \{X_{\xi_1}, \dots, X_{\xi_l}\} = \{v_1, \dots, v_{n-l}, X_{\xi_1 m}, \dots, X_{\xi_l m}\}$$

is a basis for $T_m M$. Now,

$\omega_m(X_{\xi_j m}, X_{\xi_k m}) = 0$ by equation (F) as T^l is compact and abelian,
$\omega_m(v_j, v_k) = \omega'_{[m]}([v_j],[v_k]) = 0$ as N' is Lagrangian and
$\omega_m(X_{\xi_j m}, v_k) = 0$ as $X_{\xi_j m} \in T_m(T^l.m) = (T_m\mu^{-1}(\tau))^\omega$ and also
$\mathrm{Im}\ \Omega(X_{\xi_1}, \ldots, X_{\xi_l}, v_1, \ldots, v_{n-l}) = \mathrm{Im}\ \Omega'(v_1, \ldots, v_{n-l}) = 0.$

Thus N is special Lagrangian.

Finally we prove a partial converse of the previous statement. Given a connected T^l-invariant Lagrangian submanifold $N \subset M$ consisting of regular points of μ on which T^l acts freely, then N lies in a level set of μ: As N is l-dimensional and T^l-invariant, its tangent spaces are spanned by the X_{ξ_j}. Now, as N is Lagrangian and μ is a moment map, we have

$$0 = \omega(X_{\xi_j}, \cdot)_{|N} = d\hat{\mu}(\xi_j)_{|N}$$

which implies that μ is constant on N, proving the claim. This argument can be extended to an open neighbourhood of N, showing that any T^l-invariant Lagrangian submanifold is a lift of a Lagrangian submanifold $N' \subset M//_\tau T^l$.

Example for a special Lagrangian fibration We return to the example with the map f that we have defined on page 24 but take the viewpoint of [7] which is more general than the computation we have done before. Consider the vector fields $\partial/\partial\alpha_j = 2i(\bar{z}_j\partial/\partial\bar{z}_j - z_j\partial/\partial z_j)$ inducing the standard T^n-action on $\mathbf{C}^n$. Then the vector fields $X_j = \partial/\partial\alpha_n - \partial/\partial\alpha_j$ generate a T^{n-1} action on $\mathbf{C}^n$. The standard symplectic form $\omega = dx \wedge dy = i/2 dz \wedge d\bar{z}$ and the standard volume form $\Omega = dz_1 \wedge \ldots \wedge dz_n$ are preserved by this action as the action is in $\mathrm{U}(n)$.

We can define the map $\mu = (f_1, \ldots, f_{n-1}) : \mathbf{C}^n \mapsto \mathbf{R}^{n-1} \simeq \mathbf{t^{n-1*}}$. And we see that μ is a moment map:

$$\begin{aligned} d\mu_j &= z_n d\bar{z}_n - z_j d\bar{z}_j + \bar{z}_n dz_n - \bar{z}_j dz_j \\ &= \frac{i}{2} 2i(-z_n d\bar{z}_n + z_j d\bar{z}_j - \bar{z}_n dz_n + \bar{z}_j dz_j) \\ &= \frac{i}{2} dz \wedge d\bar{z} \left(\frac{\partial}{\partial\alpha_n} - \frac{\partial}{\partial\alpha_j} \right) = \iota_{X_j}\omega \end{aligned}$$

As all of the f_j are invariant under the group action and T^{n-1} is abelian, μ is Ad^*-equivariant. Recall now that $\{f_n, f_j\} = 0$ for $1 \leq j \leq n$. Thus f_n descends to a map $\bar{f}_n$ on reduced spaces $M//_\tau T^{n-1}$ and preimages $\bar{f}_n^{-1}(v)$ for regular values v are Lagrangian submanifolds. The form induced by Ω on the reduced manifold is

$$\begin{aligned}
\Omega' &= \iota_{X_1 \dots X_{n-1}} \Omega \\
&= dz_1 \wedge \dots \wedge dz_n \left[2i \left(\bar{z}_n \frac{\partial}{\partial \bar{z}_n} - z_n \frac{\partial}{\partial z_n} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + z_1 \frac{\partial}{\partial z_1} \right), \right. \\
&\qquad \left. \dots, 2i \left(\bar{z}_n \frac{\partial}{\partial \bar{z}_n} - z_n \frac{\partial}{\partial z_n} - \bar{z}_{n-1} \frac{\partial}{\partial \bar{z}_{n-1}} + z_{n-1} \frac{\partial}{\partial z_{n-1}} \right) \right] \\
&= (2i)^{n-1} dz_1 \wedge \dots \wedge dz_n \left[z_1 \frac{\partial}{\partial z_1} - z_n \frac{\partial}{\partial z_n}, \dots, z_{n-1} \frac{\partial}{\partial z_{n-1}} - z_n \frac{\partial}{\partial z_n} \right] \\
&= c \sum_{j=1}^{n} z_1 \dots \widehat{z_j} \dots z_n dz_j \qquad \text{for some constant } c.
\end{aligned}$$

However, we know that for some regular value v of $\bar{f}_n$ and $[m] \in \bar{f}_n^{-1}(v)$, $d_{[m]}\bar{f}_n(T_{[m]}\bar{f}^{-1}(v)) = 0$. As we have seen that

$$df_n = \sum_{j=1}^{n} (i^n z_1 \dots \widehat{z_j} \dots z_n dz_j + \overline{i^n} \bar{z}_1 \dots \widehat{\bar{z}_j} \dots \bar{z}_n d\bar{z}_j)$$

this implies that

$$\Omega'_{|T_{[m]}\bar{f}_n^{-1}(v)} = 0,$$

proving that the fibres of $\bar{f}_n$ are special Lagrangian. Using the facts from above, we know that these fibres lift to special Lagrangian submanifolds in level sets of μ in M and thus $f = (\mu, f_n)$ is a special Lagrangian fibration.

END Now we have reached the topic of special Lagrangian fibrations and we have seen an example as promised. This topic reaches into the area of mirror symmetry and string theory as mentioned in the introduction. A rather non-rigourous but very readable account of the physics in string theory and its relation to Calabi-Yau manifolds and mirror symmetry is given in [6].

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