SOMETHING ON SPIN STRUCTURES

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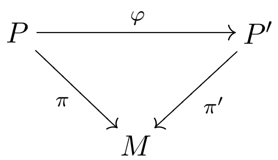
ABSTRACT

This will give a brief introduction to spin structures on vector bundles to pave the way for the definition and introduction of Dirac operators. The presentation is mainly a brief summary of that given in [1].

1 PRINCIPAL G-BUNDLES

DEFINITION Let M be a manifold, $\pi : P \to M$ be a bundle over M and G be a Lie group. If there is a right action of G on P preserving fibres and acting freely on each fibre, $\pi : P \to M$ is called a *principal G-bundle*. In particular, a principal G-bundle is locally of the form $U \times G \to U$.

EQUIVALENCE Given two principal G-bundles $\pi : P \to M$ and $\pi' : P' \to M$ over the same base space M, we say that these bundles are called *equivalent* if there exists a diffeomorphism $\varphi : P \to P'$ such that $\pi = \pi'\varphi$ (a bundle isomorphism) that is compatible with the group action, i.e. $\varphi(x.g) = \varphi(x).g$ for all $x \in M$ and $g \in G$. This gives an equivalence relation on the set of principal G-bundles over M and the set of equivalence classes will be denoted $\mathrm{Prin}_G(M)$.



EXAMPLE: COVERING SPACES Let $\pi : \tilde{M} \to M$ be a normal covering space, i.e. the group $G(M)$ of deck transformations acts transitively on each $\pi^{-1}(x)$. This is a principal $G(M)$-bundle where $G(M)$ is equipped with the discrete topology.

In particular, if $\pi : \tilde{M} \to M$ is a 2-sheeted covering, we have a $G(M) = \mathbf{Z}_2$-action on $\tilde{M}$ by the interchanging of sheets. We get that $\mathrm{Prin}_{\mathbf{Z}_2}(M) \simeq \mathrm{Cov}_2(M)$. Similarly the universal covering of S^1, $\pi : \mathbf{R} \to \mathrm{S}^1$, is a principal $\mathbf{Z}$-bundle.

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EXAMPLE: HOPF FIBRATION A more exciting example is given by the Hopf fibration $\pi : \mathrm{S}^{2n+1} \to \mathbf{CP}^n$ with fibre S^1: it is a principal S^1-bundle.

EXAMPLE: BUNDLES OF BASES For a different kind of example, consider $\pi : E \to M$, a n-dimensional real vector bundle. Then the *bundle of bases* is defined to have as fibre at x the set of all bases for the fibre E_x of the vector bundle. This bundle is denoted $\mathrm{P}_{\mathrm{GL}}(E)$ and it is a principal GL_n-bundle.

Similarly we can define the *bundle* $\mathrm{P}_{\mathrm{O}_n}(E)$ *of orthonormal bases for* E which is a principal O_n-bundle. Finally, for oriented E, we can define the *bundle* $\mathrm{P}_{\mathrm{SO}_n}(E)$ *of oriented orthonormal bases for* E which is a principal SO_n-bundle.

ASIDE: ČECH-COHOMOLOGY A knowledge of Čech-cohomology helps when looking at the equivalence classes of principal G-bundles. This is due to the fact that the compatibility condition for the change of local trivialisation on bundles and the coboundary condition in Čech-cohomology are related. Working through the technical details gives the useful relation $\mathrm{Prin}_G(M) \simeq \check{H}^1(M, \mathcal{G})$ where $\mathcal{G}$ denotes the sheaf of G-valued germs.

A deduction of this particular relation can be found in [1, p. 371f] and a comprehensive treatment of Čech-cohomology can usually be found in textbooks on topics dealing with sheaves, e.g. algebraic geometry.

2 ORIENTABILITY

Given a riemannian manifold M and $\pi : E \to M$, a n-dimensional real vector bundle, we can try to find out whether this vector bundle admits an orientation on each fibre E_x that is continuously varying with x.

To formalise this problem, we consider the bundle of orthonormal bases $\mathrm{P}_{\mathrm{O}_n}(E)$ as in the example above. Now form the quotient $\mathrm{Or}(E) = \mathrm{P}_{\mathrm{O}_n}(E)/\mathrm{SO}_n$, where two elements of $\mathrm{P}_{\mathrm{O}_n}(E)_x$ are identified if they are related by an element of SO_n, i.e. if they have the same orientation. Thus the fibre at x of $\mathrm{Or}(E)$ consists of the orientations of E_x and consequently $\mathrm{Or}(E)$ is called the *bundle of orientations in E*.

$\mathrm{Or}(E)$ is a 2-sheeted covering space of M. E is orientable if and only if $\mathrm{Or}(E)$ is a trivial bundle as only this ensures that there are two distinct continuously varying orientations on each connected component. Thus, if E is orientable, there is a one-to-one correspondence between orientations and elements of $H^0(M, \mathbf{Z}_2)$.

EXAMPLE As an easy example, compare a Möbius strip M and a cylinder C, both of which can be thought of as one-dimensional real vector bundles over S^1. While $\mathrm{Or}(C) \simeq \mathrm{S}^1 \times \mathbf{Z}_2$, i.e. a trivial bundle, $\mathrm{Or}(M)$ is the non-trivial double-covering of S^1.

ISOMORPHISM We can see that $\mathrm{Cov}_2(M) \simeq \mathrm{Hom}(\pi_1(M), \mathbf{Z}_2)$ as there is a one-to-one correspondence between 2-sheeted coverings of M and index-2-subgroups of $\pi_1(M)$. As we are looking at homomorphisms to $\mathbf{Z}_2$, we can exchange $\pi_1(M)$ by its abelianisation $H_1(M)$, and using Poincaré duality we get $\mathrm{Hom}(\pi_1(M), \mathbf{Z}_2) \simeq \mathrm{Hom}(H_1(M), \mathbf{Z}_2) \simeq H^1(M, \mathbf{Z}_2)$. Thus, we have established an isomorphism

$$\mathrm{Cov}_2(M) \simeq H^1(M, \mathbf{Z}_2).$$

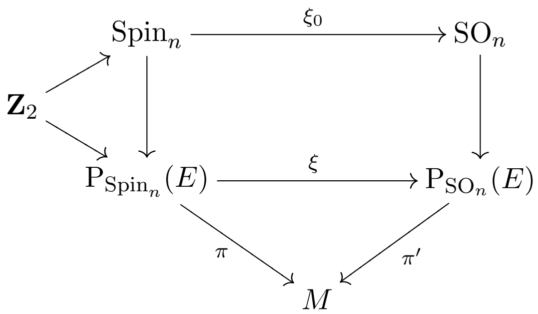
This isomorphism natural and it is the same as the one presented in the paragraph on Čech-cohomology above.

FIRST STIEFEL-WHITNEY CLASS As every vector bundle E as above gives an element $\mathrm{Or}(E)$ of $\mathrm{Cov}_2(M)$, it also gives an element $w_1(E)$ in $H^1(M, \mathbf{Z}_2)$ via the isomorphism we just established. $w_1(E)$ is called the *first Stiefel-Whitney class of* E and E is orientable if and only if $w_1(E) = 0$. We can think of $w_1(E)$ as the obstruction to the orientability of E.

MOTIVATION Another way to think about choosing an orientation for an orientable vector bundle E is that it is equivalent to choosing a principal SO_n bundle $\mathrm{P}_{\mathrm{SO}_n}(E) \subset \mathrm{P}_{\mathrm{O}_n}(E)$ for E. In particular this means, we are able to make the structure group for E connected for orientable E. This may motivate the next step in which we try to make the structure group not only connected but also simply connected.

3 SPIN STRUCTURES

DEFINITION For $n \geq 3$, a *spin structure* on an oriented n-dimensional real vector bundle $\pi : E \to M$ is a principal Spin_n-bundle $\mathrm{P}_{\mathrm{Spin}_n}(E)$ with a 2-sheeted covering $\xi : \mathrm{P}_{\mathrm{Spin}_n}(E) \to \mathrm{P}_{\mathrm{SO}_n}(E)$ that is compatible with the Spin_n-action, i.e. $\xi(x.g) = \xi(x).\xi_0(g)$, where ξ_0 is the universal cover of SO_n. This definition is best summarised by the diagram:



Re-phrasing the above definition, we get that there is a one-to-one correspondence between spin structures on E and 2-sheeted coverings of $\mathrm{P}_{\mathrm{SO}_n}(E)$

that are non-trivial on fibres. By the isomorphism of the previous section, they are also in one-to-one correspondence with elements of $H^1(\mathrm{P}_{\mathrm{SO}_n}(E), \mathbf{Z}_2)$ that restrict to non-zero elements on fibres of $\mathrm{P}_{\mathrm{SO}_n}(E)$.

SECOND STIEFEL-WHITNEY CLASS AND FURTHER RESULTS With further work the *second Stiefel-Whitney class* $w_2(E)$ can be defined. This requires a closer look at the cohomology. One way to do this, is to recall that an oriented vector bundle gives us a principal SO_n-bundle on M and use the isomorphism $\mathrm{Prin}_{\mathrm{SO}_n}(M) \simeq H^1(M, \mathrm{SO}_n)$ to consider E as an element of $H^1(M, \mathrm{SO}_n)$. Now, the second Stiefel-Whitney class can be defined as the coboundary map $w_2 : H^1(M, \mathrm{SO}_n) \to H^2(M, \mathbf{Z}_2)$ that is induced (in a non-obvious way, see [1, p. 373]) by the short exact sequence $0 \to \mathbf{Z}_2 \to \mathrm{Spin}_n \to \mathrm{SO}_n \to 0$.

We then get results corresponding to those for w_1: A spin structure exists on E if and only if $w_2(E) = 0$ and if $w_2(E) = 0$, then the spin-structures on E are in one-to-one correspondence with $H^1(M, \mathbf{Z}_2)$. It can also be shown, that once we choose a spin structure on E for a given riemannian metric on E, this determines spin structures with respect to all other riemannian metrics on E.

SPIN MANIFOLDS An oriented riemannian manifold with a spin structure on its tangent bundle is called a *spin manifold*. As an example, we can consider the manifold SO_n of dimension $N = \frac{n(n-1)}{2}$. We know that $H_1(\mathrm{SO}_n) \simeq \pi_1(\mathrm{SO}_n) \simeq \mathbf{Z}_2$ and thus $H^1(\mathrm{SO}_n, \mathbf{Z}_2) \simeq \mathrm{Hom}(H_1(\mathrm{SO}_n), \mathbf{Z}_2) \simeq \mathbf{Z}_2$. Hence there are two distinct spin structures on SO_n. As SO_n is a Lie group, it is parallelisable, and thus $\mathrm{P}_{\mathrm{SO}_N}(\mathrm{SO}_n) = \mathrm{SO}_n \times \mathrm{SO}_N$. The two 2-covers of $\mathrm{P}_{\mathrm{SO}_N}(\mathrm{SO}_n)$ are then given by

$$\widetilde{\mathrm{P}_{\mathrm{SO}_N}(\mathrm{SO}_n)} = \mathrm{SO}_n \times \mathrm{Spin}_N \quad \text{and} \quad \widehat{\mathrm{P}_{\mathrm{SO}_N}(\mathrm{SO}_n)} = (\mathrm{Spin}_n \times \mathrm{Spin}_N)/\mathbf{Z}_2$$

where $\mathbf{Z}_2$ identifies (p, q) and $(-p, -q)$ in $\mathrm{Spin}_n \times \mathrm{Spin}_N$.

4 CLIFFORD BUNDLES

THE ASSOCIATED BUNDLE Let M, F be manifolds, G a Lie group, $\pi : P \to M$ a principal G-bundle and $\varrho : G \to \mathrm{Homeo}(F)$ a continuous group homomorphism. Then ϱ allows us to define a free left action of G on $P \times F$ by

$$g.(p, f) = \left(p.g^{-1}, \varrho(g)(f)\right)$$

which we use to define the quotient $P \times_\varrho F = (P \times F)/\sim$ where $(p, f) \sim (p', f')$ if and only if $(p, f) = g.(p', f')$ for some $g \in G$. The map $P \times F \xrightarrow{p_1} P \xrightarrow{\pi} M$ descends to the quotient as $\pi p_1(g.(p, f)) = \pi p_1(p.g^{-1}, \varrho(g)(f)) = \pi(p.g^{-1}) = \pi(p)$ as fibres are invariant under the action of G, making $P \times_\varrho F$ into a fibre bundle with fibre F over M. $P \times_\varrho F$ is called *the bundle associated to P by ϱ.*

REPRESENTATION OF SO_n Recall the standard representation of SO_n on $\mathbf{R}^n$:

$$\varrho_n : \mathrm{SO}_n \longrightarrow \mathrm{Aut}(\mathbf{R}^n) \qquad o \longmapsto (\varrho_n^o : v \mapsto o(v))$$

Using functoriality of Cl as shown in [4, §4], this induces a representation of SO_n on Cl_n:

$$cl(\varrho_n) : \mathrm{SO}_n \longrightarrow \mathrm{Aut}(Cl_n) \qquad o \longmapsto (cl(\varrho_n)^o : x \mapsto \tilde{o}(x))$$

DEFINITION Using the objects of the previous paragraphs we can now define the *Clifford bundle* for an oriented n-dimensional riemannian vector bundle E:

$$Cl(E) = \mathrm{P}_{\mathrm{SO}_n}(E) \times_{cl(\varrho_n)} Cl_n$$

Note that the fibres of $Cl(E)$ are $Cl(E)_x = Cl(E_x, q_x)$ where q_x is the riemannian metric at x.

By virtue of the naturality of most of the properties of Clifford algebras, such as the grading, these properties exist similarly for Clifford bundles.

5 OUTLOOK

The next step will be to define and discuss spinor bundles using Clifford modules and the Dirac operator as it is done in [3] and [1, II].

It seems that the subject of spin structures can be approached from many different angles. Some authors prefer to emphasise the representation theory [2], while others prefer to focus on cohomology. It seems that further study of characteristic classes will be helpful; [5] was recommended for this purpose in general and [6] was recommended for Seiberg-Witten theory.

REFERENCES

[1] H. BLAINE LAWSON, JR. AND MARIE-LOUISE MICHELSOHN. Spin Geometry. Princeton University Press, Princeton, 1989. QA401.L2

[2] DIETMAR SALAMON. Chapter 4 of a forthcoming book on Spin Geometry.

[3] JOHN HOPKINSON. Talk on Spin groups and Dirac operators in the Riemannian Geometry seminar, Spring 2001.

[4] SVEN-S. PORST. Bits and Pieces on Clifford Algebras, Spring 2001; http://homepage.mac.com/ssp/studium/2001spring/Clifford.pdf

[5] JOHN WILLARD MILNOR AND JAMES STASHEFF. Characteristic Classes, Princeton University Press, 1974. QA612.M4

[6] JOHN W. MORGAN. The Seiberg-Witten equations and applications to the topology of smooth four manifolds, Princeton University Press, 1996. QA612.M6