ABSTRACT

This will give a brief introduction to spin structures on vector bundles to pave the way for the definition and introduction of Dirac operators. The presentation is mainly a brief summary of that given in [1].

1 PRINCIPAL $G$-BUNDLES

DEFINITION Let $M$ be a manifold, $\pi : P \to M$ be a bundle over $M$ and $G$ be a Lie group. If there is a right action of $G$ on $P$ preserving fibres and acting freely on each fibre, $\pi : P \to M$ is called a principal $G$-bundle. In particular, a principal $G$-bundle is locally of the form $U \times G \to U$.

EQUIVALENCE Given two principal $G$-bundles $\pi : P \to M$ and $\pi' : P' \to M$ over the same base space $M$, we say that these bundles are called equivalent if there exists a diffeomorphism $\varphi : P \to P'$ such that $\pi = \pi' \varphi$ (a bundle isomorphism) that is compatible with the group action, i.e. $\varphi(x.g) = \varphi(x).g$ for all $x \in M$ and $g \in G$.

This gives an equivalence relation on the set of principal $G$-bundles over $M$ and the set of equivalence classes will be denoted $\text{Prin}_G(M)$.

EXAMPLE: COVERING SPACES Let $\pi : \tilde{M} \to M$ be a normal covering space, i.e. the group $G(M)$ of deck transformations acts transitively on each $\pi^{-1}(x)$. This is a principal $G(M)$-bundle where $G(M)$ is equipped with the discrete topology.

In particular, if $\pi : \tilde{M} \to M$ is a 2-sheeted covering, we have a $G(M) = \mathbb{Z}_2$-action on $\tilde{M}$ by the interchanging of sheets. We get that $\text{Prin}_{\mathbb{Z}_2}(M) \simeq \text{Cov}_2(M)$. Similarly the universal covering of $S^1$, $\pi : \mathbb{R} \to S^1$, is a principal $\mathbb{Z}$-bundle.
EXAMPLE: HOPF FIBRATION  A more exciting example is given by the Hopf fibration \( \pi: S^{2n+1} \to \mathbb{C}P^n \) with fibre \( S^1 \): it is a principal \( S^1 \)-bundle.

EXAMPLE: BUNDLES OF BASES  For a different kind of example, consider \( \pi: E \to M \), a \( n \)-dimensional real vector bundle. Then the bundle of bases is defined to have as fibre at \( x \) the set of all bases for the fibre \( E_x \) of the vector bundle. This bundle is denoted \( P_{\text{GL}}(E) \) and it is a principal \( \text{GL}_n \)-bundle.

Similarly we can define the bundle \( P_{\text{O}_n}(E) \) of orthonormal bases for \( E \) which is a principal \( \text{O}_n \)-bundle. Finally, for oriented \( E \), we can define the bundle \( P_{\text{SO}_n}(E) \) of oriented orthonormal bases for \( E \) which is a principal \( \text{SO}_n \)-bundle.

ASIDE: ČECH-COHOMOLOGY  A knowledge of Čech-cohomology helps when looking at the equivalence classes of principal \( G \)-bundles. This is due to the fact that the compatibility condition for the change of local trivialisation on bundles and the coboundary condition in Čech-cohomology are related. Working through the technical details gives the useful relation \( \text{Prin}_G(M) \simeq H^1(M, G) \) where \( G \) denotes the sheaf of \( G \)-valued germs.

A deduction of this particular relation can be found in [1, p. 371f] and a comprehensive treatment of Čech-cohomology can usually be found in textbooks on topics dealing with sheaves, e.g. algebraic geometry.

2 ORIENTABILITY

Given a riemannian manifold \( M \) and \( \pi: E \to M \), a \( n \)-dimensional real vector bundle, we can try to find out whether this vector bundle admits an orientation on each fibre \( E_x \) that is continuously varying with \( x \).

To formalise this problem, we consider the bundle of orthonormal bases \( P_{\text{O}_n}(E) \) as in the example above. Now form the quotient \( \text{Or}(E) = P_{\text{O}_n}(E)/\text{SO}_n \), where two elements of \( P_{\text{O}_n}(E)_x \) are identified if they are related by an element of \( \text{SO}_n \), i.e. if they have the same orientation. Thus the fibre at \( x \) of \( \text{Or}(E) \) consists of the orientations of \( E_x \) and consequently \( \text{Or}(E) \) is called the bundle of orientations in \( E \).

\( \text{Or}(E) \) is a 2-sheeted covering space of \( M \). \( E \) is orientable if and only if \( \text{Or}(E) \) is a trivial bundle as only this ensures that there are two distinct continuously varying orientations on each connected component. Thus, if \( E \) is orientable, there is a one-to-one correspondence between orientations and elements of \( H^0(M, \mathbb{Z}_2) \).

EXAMPLE  As an easy example, compare a Möbius strip \( M \) and a cylinder \( C \), both of which can be thought of as one-dimensional real vector bundles over \( S^1 \). While \( \text{Or}(C) \simeq S^1 \times \mathbb{Z}_2 \), i.e. a trivial bundle, \( \text{Or}(M) \) is the non-trivial double-covering of \( S^1 \).
isomorphism. We can see that \( \text{Cov}_2(M) \simeq \text{Hom}(\pi_1(M), \mathbb{Z}_2) \) as there is a one-to-one correspondence between 2-sheeted coverings of \( M \) and index-2-subgroups of \( \pi_1(M) \). As we are looking at homomorphisms to \( \mathbb{Z}_2 \), we can exchange \( \pi_1(M) \) by its abelianisation \( H_1(M) \), and using Poincaré duality we get \( \text{Hom}(\pi_1(M), \mathbb{Z}_2) \simeq \text{Hom}(H_1(M), \mathbb{Z}_2) \simeq H^1(M, \mathbb{Z}_2) \). Thus, we have established an isomorphism

\[
\text{Cov}_2(M) \simeq H^1(M, \mathbb{Z}_2).
\]

This isomorphism natural and it is the same as the one presented in the paragraph on Čech-cohomology above.

**First Stiefel-Whitney Class**

As every vector bundle \( E \) as above gives an element \( \text{Or}(E) \) of \( \text{Cov}_2(M) \), it also gives an element \( w_1(E) \) in \( H^1(M, \mathbb{Z}_2) \) via the isomorphism we just established. \( w_1(E) \) is called the first Stiefel-Whitney class of \( E \) and \( E \) is orientable if and only if \( w_1(E) = 0 \). We can think of \( w_1(E) \) as the obstruction to the orientability of \( E \).

**Motivation**

Another way to think about choosing an orientation for an orientable vector bundle \( E \) is that it is equivalent to choosing a principal \( \text{SO}_n \) bundle \( P_{\text{SO}_n}(E) \subset P_{O_n}(E) \) for \( E \). In particular this means, we are able to make the structure group for \( E \) connected for orientable \( E \). This may motivate the next step in which we try to make the structure group not only connected but also simply connected.

### 3 Spin Structures

**Definition**

For \( n \geq 3 \), a *spin structure* on an oriented \( n \)-dimensional real vector bundle \( \pi : E \to M \) is a principal \( \text{Spin}_n \)-bundle \( P_{\text{Spin}_n}(E) \) with a 2-sheeted covering \( \xi : P_{\text{Spin}_n}(E) \to P_{\text{SO}_n}(E) \) that is compatible with the \( \text{Spin}_n \)-action, i.e. \( \xi(x,g) = \xi(x)\xi_0(g) \), where \( \xi_0 \) is the universal cover of \( \text{SO}_n \).

This definition is best summarised by the diagram:

\[
\begin{array}{ccc}
\mathbb{Z}_2 & \xrightarrow{\xi_0} & \text{SO}_n \\
\downarrow & & \downarrow \\
P_{\text{Spin}_n}(E) & \xrightarrow{\xi} & P_{\text{SO}_n}(E) \\
\downarrow \pi & & \downarrow \pi' \\
M & & \\
\end{array}
\]

Re-phrasing the above definition, we get that there is a one-to-one correspondence between spin structures on \( E \) and 2-sheeted coverings of \( P_{\text{SO}_n}(E) \)
that are non-trivial on fibres. By the isomorphism of the previous section, they are also in one-to-one correspondence with elements of $H^1(\text{Prin}_{SO_n}(E), \mathbb{Z}_2)$ that restrict to non-zero elements on fibres of $\text{Prin}_{SO_n}(E)$.

**SECOND STIEFEL-WHITNEY CLASS AND FURTHER RESULTS** With further work the second Stiefel-Whitney class $w_2(E)$ can be defined. This requires a closer look at the cohomology. One way to do this, is to recall that an oriented vector bundle gives us a principal $SO_n$-bundle on $M$. And use the isomorphism $\text{Prin}_{SO_n}(M) \simeq H^1(M, SO_n)$ to consider $E$ as an element of $H^1(M, SO_n)$.

Now, the second Stiefel-Whitney class can be defined as the coboundary map $w_2 : H^1(M, SO_n) \to H^2(M, \mathbb{Z}_2)$ that is induced (in a non-obvious way, see [11 p. 373]) by the short exact sequence $0 \to \mathbb{Z}_2 \to \text{Spin}_n \to SO_n \to 0$.

We then get results corresponding to those for $w_1$: A spin structure exists on $E$ if and only if $w_2(E) = 0$ and if $w_2(E) = 0$, then the spin-structures on $E$ are in one-to-one correspondence with $H^1(M, \mathbb{Z}_2)$. It can also be shown, that once we choose a spin structure on $E$ for a given riemannian metric on $E$, this determines spin structures with respect to all other riemannian metrics on $E$.

**SPIN MANIFOLDS** An oriented riemannian manifold with a spin structure on its tangent bundle is called a *spin manifold*. As an example, we can consider the manifold $SO_n$ of dimension $N = \frac{n(n-1)}{2}$. We know that $H_1(SO_n) \simeq \pi_1(SO_n) \simeq \mathbb{Z}_2$ and thus $H^1(SO_n, \mathbb{Z}_2) \simeq \text{Hom}(H_1(SO_n), \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Hence there are two distinct spin structures on $SO_n$. As $SO_n$ is a Lie group, it is parallelisable, and thus $\text{Prin}_{SO_n}(SO_n) = SO_n \times SO_N$. The two 2-covers of $\text{Prin}_{SO_n}(SO_n)$ are then given by

$$\text{Prin}_{SO_n}(SO_n) = SO_n \times \text{Spin}_N \quad \text{and} \quad \text{Prin}_{SO_n}(SO_n) = (\text{Spin}_n \times \text{Spin}_N)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ identifies $(p, q)$ and $(-p, -q)$ in $\text{Spin}_n \times \text{Spin}_N$.

### 4 CLIFFORD BUNDLES

**THE ASSOCIATED BUNDLE** Let $M$, $F$ be manifolds, $G$ a Lie group, $\pi : P \to M$ a principal $G$-bundle and $\varrho : G \to \text{Homeo}(F)$ a continuous group homomorphism. Then $\varrho$ allows us to define a free left action of $G$ on $P \times F$ by

$$g.(p, f) = \left(p, g^{-1}(\varrho(g)(f))\right)$$

which we use to define the quotient $P \times_\varrho F = (P \times F) / \sim$ where $(p, f) \sim (p', f')$ if and only if $(p, f) = g.(p', f')$ for some $g \in G$. The map $P \times F \xrightarrow{\pi_1} P \xrightarrow{\pi_2} M$ descends to the quotient as $\pi_1(g.(p, f)) = \pi_1(p, g^{-1}(\varrho(g)(f))) = \pi(p, g^{-1}) = \pi(p)$ as fibres are invariant under the action of $G$, making $P \times_\varrho F$ into a fibre bundle with fibre $F$ over $M$. $P \times_\varrho F$ is called the *bundle associated to $P$ by $\varrho$*. 

4
Recall the standard representation of $SO_n$ on $\mathbb{R}^n$:

$$\varrho_n : SO_n \rightarrow \text{Aut}(\mathbb{R}^n) \quad o \mapsto (\varrho_n^o : v \mapsto o(v))$$

Using functoriality of $Cl$ as shown in [4, §4], this induces a representation of $SO_n$ on $Cl_n$:

$$cl(\varrho_n) : SO_n \rightarrow \text{Aut}(Cl_n) \quad o \mapsto (cl(\varrho_n)^o : x \mapsto \tilde{o}(x))$$

Definition Using the objects of the previous paragraphs we can now define the Clifford bundle for an oriented $n$-dimensional riemannian vector bundle $E$:

$$Cl(E) = P_{SO_n}(E) \times_{cl(\varrho_n)} Cl_n$$

Note that the fibres of $Cl(E)$ are $Cl(E)_x = Cl(E_x, q_x)$ where $q_x$ is the riemannian metric at $x$.

By virtue of the naturality of most of the properties of Clifford algebras, such as the grading, these properties exist similarly for Clifford bundles.

5 Outlook

The next step will be to define and discuss spinor bundles using Clifford modules and the Dirac operator as it is done in [3] and [1, II].

It seems that the subject of spin structures can be approached from many different angles. Some authors prefer to emphasise the representation theory [2], while others prefer to focus on cohomology. It seems that further study of characteristic classes will be helpful; [5] was recommended for this purpose in general and [6] was recommended for Seiberg-Witten theory.

References


